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THE MODERN MATHEMATICAL SERIES

LUCIEN AUGUSTUS WAIT . . . GENERAL EDITOR

(SENIOR PROFESSOR OF MATHEMATICS IN CORNELL UNIVERSITY)

THE MODERN MATHEMATICAL SERIES.

LUCIEN AUGUSTUS WAIT,
(Senior Professor of Mathematics in Cornell University,)
GENERAL EDITOR.

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The Analytic Geometry, Differential Calculus, and Integral Calculus (published in September of 1898) were written primarily to meet the needs of college students pursuing courses in Engineering and Architecture; accordingly, practical problems, in illustration of general principles under discussion, play an important part in each book.

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DIFFERENTIAL AND INTEGRAL CALCULUS

BY

VIRGIL SNYDER, PH.D. (GÖTTINGEN)
AND
JOHN IRWIN HUTCHINSON, PH.D. (CHICAGO)
OF CORNELL UNIVERSITY

* * *

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PREFACE

The favorable reception accorded the two volumes on the Calculus in this series shows that they have been serviceable in supplying a real need. A general demand has arisen for a similar treatment of the subjects in briefer form, suitable for use in shorter and more elementary courses. Accordingly, in response to numerous requests and suggestions, the present volume has been prepared.

The part on the Differential Calculus is of essentially the same character as the former separate volume (which will be referred to in the text as D. C.), but the range of topics is restricted; various theorems have been put in less abstract form, and fewer alternative proofs have been given. The chapter on the expansion of functions has been so arranged that the remainder theorem may be omitted without marring the continuity of the subject. In the treatment of functions of two independent variables no use is made of an auxiliary variable.

The characteristic features of the larger book are retained. Some of these are as follows: —

1. The derivative is presented rigorously as a limit.

2. The process of differentiation is so arranged as to give the x-derivative of a function of u, in which u is a function of x; the resulting type forms being printed in full-face letters in the text and collected for reference at the end of the chapter.

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3. Maxima and minima are discussed as the turning values in the variation of a function, with complete graphical representation.

4. The notions of rates and differentials are so presented as to grow naturally out of the idea of a derivative, and are not introduced until the student has become familiar with the process of finding the derivative and with its use in studying the variation of a function.

5. The related theories of inflexions, curvature, and asymptotes receive direct and comprehensive treatment.

The part on the Integral Calculus has been written entirely anew.

The first five chapters discuss the ordinary methods of integration. The aim has been to make clear the *rationale* of each process, and to encourage the students to become independent of formulas.

The method of reduction has been put in the simplest possible form ; in the solution of problems students need make no use of formulas of reduction.

In the resolution of rational fractions into simpler ones, care has been taken to show the logical basis of the usual assumptions.

The rationalization of a differential containing the square root of a quadratic expression has been treated much more fully than usual. The problem is interpreted geometrically as equivalent to the rational expression of the coördinates of a variable point on a conic in terms of a varying parameter. This makes clear how the required transformations are suggested and puts the subject in a more attractive form.

Special care has been taken in presenting the subject of integration regarded as a summation so as to combine rigor and simplicity. The ordinary cases of discontinuity, either of the integrand or of the variable of integration, are included in the discussion.

In deriving the formula for length of arc, the definition of such length is given as the limit of the sum of chords, a definition which readily expresses itself, by the use of the mean-value theorem, in the form of a definite integral.

The exercises, which are new throughout the book, are carefully graded. Numerous illustrative examples are worked out in the text, and are accompanied by various suggestions and remarks relating to both theory and practice.

The authors gratefully acknowledge their indebtedness to their colleague, Professor James McMahon, for permission to make free use of McMahon and Snyder's Differential Calculus, for a number of valuable suggestions, and for assistance in reading portions of the manuscript and proof.

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DIFFERENTIAL CALCULUS

CHAPTER I

FUNDAMENTAL PRINCIPLES

1. Elementary definitions. A *constant* number is one that retains the same value throughout an investigation in which it occurs. A *variable* number is one that changes from one value to another during an investigation. When the variation of a number can be assigned at will, the variable is called *independent;* when the value of one number is determined by that of another, the former is called a *dependent* variable. The dependent variable is called a *function* of the independent variable.

E.g., $3x^2$, $4\sqrt{x-1}$, $\cos x$, are all functions of x.

Functions of one variable x will be denoted by the symbols $f(x)$, $\phi(x)$, $\cdots$; similarly, if z be a function of two variables x, y, it will be denoted by such expressions as

$$z=f(x, y),\quad z=F(x, y)\cdots.$$

When a variable approaches a constant in such a way that the difference between the variable and the constant may become and remain smaller than any fixed number, previously assigned, the constant is called the *limit* of the variable.

There is nothing in this definition which requires a variable to attain the value of its limit, or not to attain it. The examples of limits met with in elementary geometry are usually of the second kind; *i.e.* the variable does not reach the limit. The limiting values of algebraic expressions are more frequently of the first kind.

E.g., the function $\frac{1}{x^2+1}$ has the limit 1 when x becomes zero; it has the limit 0 when x becomes infinite. The function $\sin x$ has the limit 0 when x becomes zero; $\tan x$ has the limit 1 when x becomes $\frac{\pi}{4}$.

EXERCISES

1. Let $\psi(x, y) = Ax + By + C$; show that $\psi(x, y) = 0$, $\psi(y, -x) = 0$ are the equations of two perpendicular lines.

2. If $f(x) = 2x\sqrt{1-x^2}$, show that $f\left(\sin\frac{x}{2}\right) = \sin x = f\left(\cos\frac{x}{2}\right)$.

3. If $\phi(x) = \frac{x-1}{x+1}$, show that $\frac{\phi(x)-\phi(y)}{1+\phi(x)\phi(y)} = \frac{x-y}{1+xy}$.

4. If $f(x) = \log\frac{1-x}{1+x}$, show that $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$.

5. Given $f(x) = \sqrt{1-x^2}$, find $f(\sqrt{1-x^2})$.

6. If $f(xy) = f(x) + f(y)$, prove that $f(1) = 0$.

7. Given $f(x+y) = f(x) + f(y)$, show that $f(0) = 0$, and that $pf(x) = f(px)$, p being any positive integer.

8. Using the same notation as in the last example, prove that $f(mx) = mf(x)$, m being any rational fraction.

2. Infinitesimals and infinites. A variable that approaches zero as a limit is an *infinitesimal*. In other words, an infinitesimal is a variable that becomes smaller than any number that can be assigned.

The reciprocal of an infinitesimal is then a variable that becomes larger than any number that can be assigned, and is called an *infinite* variable.

E.g., the number $(\frac{1}{2})^n$ is an infinitesimal when n is taken larger and larger; and its reciprocal 2^n is an infinite variable.

From the definitions of the words "limit" and "infinitesimal" the following useful corollaries are immediate inferences.

Cor. 1. The difference between a variable and its limit is an infinitesimal variable.

Cor. 2. Conversely, if the difference between a constant and a variable be an infinitesimal, then the constant is the limit of the variable.

For convenience, the symbol $\doteq$ will be used to indicate that a variable approaches a constant as a limit; thus the symbolic form $x \doteq a$ is to be read "the variable x approaches the constant a as a limit."

The special form $x \doteq \infty$ is read "x becomes infinite."

The corollaries just mentioned may accordingly be symbolically stated thus:

1. If $x \doteq a$, then $x = a + \alpha$, wherein $\alpha \doteq 0$;
2. If $x = a + \alpha$, and $\alpha \doteq 0$, then $x \doteq a$.

It will appear that the chief use of Cor. 1 is to convert given limit relations into the form of ordinary equations, so that they may be combined or transformed by the laws governing the equality of numbers; and then Cor. 2 will serve to express the result in the original form of a limit relation.

In all cases, whether a variable actually becomes equal to its limit or not, the important property is that their difference is an infinitesimal. An infinitesimal is not necessarily in all stages of its history a small number. Its essence lies in its power of decreasing numerically, having zero for its limit, and not in the smallness of any of the constant values it may pass through. It is frequently defined as an "infinitely small quantity," but this expression should be interpreted in the above sense. Thus a constant number, however small it may be, is not an infinitesimal.

3. Fundamental theorems concerning infinitesimals and limits in general. The following theorems are useful in the processes of the calculus; the first three relate to infinitesimals, the last four to limits in general.

Theorem 1. The product of an infinitesimal α by any finite constant k is an infinitesimal;

i.e., if $$\alpha \doteq 0,$$

then $$k\alpha \doteq 0.$$

For, let c be any assigned number. Then, by hypothesis, α can become less than $\frac{c}{k}$; hence $k\alpha$ can become less than c, the arbitrary, assigned number, and is, therefore, infinitesimal.

Theorem 2. The algebraic sum of any finite number n of infinitesimals is an infinitesimal;

i.e., if $$\alpha \doteq 0,\ \beta \doteq 0,\ \cdots,$$

then $$\alpha + \beta + \cdots \doteq 0.$$

For the sum of the n variables does not at any stage numerically exceed n times the largest of them, but this product is an infinitesimal by theorem 1; hence the sum of the n variables is either an infinitesimal or zero.

Note. The sum of an infinite number of infinitesimals may be infinitesimal, finite, or infinite, according to circumstances.

E.g., if a be a finite constant, and if n be a variable that becomes infinite; then $\frac{a}{n^2}$, $\frac{a}{n}$, $\frac{a}{n^{\frac{1}{2}}}$, are all infinitesimal variables; but

$$\frac{a}{n^2} + \frac{a}{n^2} + \cdots \text{ to } n \text{ terms} = \frac{a}{n}, \text{ which is infinitesimal,}$$

while $$\frac{a}{n} + \frac{a}{n} + \cdots \text{ to } n \text{ terms} = a, \text{ which is finite,}$$

and $$\frac{a}{n^{\frac{1}{2}}} + \frac{a}{n^{\frac{1}{2}}} + \cdots \text{ to } n \text{ terms} = an^{\frac{1}{2}}, \text{ which is infinite.}$$

Theorem 3. The product of two or more infinitesimals is an infinitesimal.

Theorem 4. If two variables x, y be always equal, and if one of them, x, approach a limit a, then the other approaches the same limit.

Theorem 5. If the sum of a finite number of variables be variable, then the limit of their sum is equal to the sum of their limits;

i.e., $$\lim(x+y+\cdots)=\lim x+\lim y+\cdots.$$

For, let $$x \doteq a, \quad y \doteq b, \quad \cdots.$$

Then $$x=a+\alpha, \quad y=b+\beta, \quad \cdots, \qquad \text{[Art. 2, Cor. 1.}$$

wherein $$\alpha \doteq 0, \quad \beta \doteq 0, \quad \cdots;$$

hence $$x+y+\cdots=(a+b+\cdots)+(\alpha+\beta+\cdots);$$

but $$\alpha+\beta+\cdots \doteq 0, \qquad \text{[Th. 2.}$$

hence, by Art. 2, Cor. 2,

$$\lim(x+y+\cdots)=a+b+\cdots=\lim x+\lim y+\cdots.$$

Theorem 6. If the product of a finite number of variables be variable, then the limit of their product is equal to the product of their limits.

Theorem 7. If the quotient of two variables x, y be variable, then the limit of their quotient is equal to the quotient of their limits, provided these limits are not both infinite, or not both zero.

4. Comparison of variables. Some of the principles just established will now be used in comparing variables with each other. The relative importance of two variables that are approaching limits is measured by the limit of their ratio.

DEFINITION. One variable α is said to be infinitesimal, infinite, or finite, in comparison with another variable x when the limit of their ratio $\alpha : x$ is zero, infinite, or finite.

In the first two cases, the phrase "infinitesimal or infinite in comparison with" is sometimes replaced by the less precise phrase "infinitely smaller or infinitely larger than." In the third case, the variables will be said to be of the same order of magnitude.

The following theorem and corollary are useful in comparing two variables:

Theorem 8. The limit of the quotient of any two variables x, y is not altered by adding to them any two numbers α, β, which are respectively infinitesimal in comparison with these variables;

i.e.,
$$\lim \frac{x+\alpha}{y+\beta} = \lim \frac{x}{y},$$

provided
$$\frac{\alpha}{x} \doteq 0, \quad \frac{\beta}{y} \doteq 0.$$

For, since
$$\frac{x+\alpha}{y+\beta} = \frac{x}{y} \cdot \frac{1+\dfrac{\alpha}{x}}{1+\dfrac{\beta}{y}},$$

it follows, by theorems 4, 6, that
$$\lim \frac{x+\alpha}{y+\beta} = \lim \frac{x}{y} \cdot \lim \frac{1+\dfrac{\alpha}{x}}{1+\dfrac{\beta}{y}};$$

but, by theorems 7, 5, and hypothesis,
$$\lim \frac{1+\dfrac{\alpha}{x}}{1+\dfrac{\beta}{y}} = 1;$$

therefore,
$$\lim \frac{x+\alpha}{y+\beta} = \lim \frac{x}{y}.$$

Cor. If the difference between two variables x, y be infinitesimal as to either, the limit of their ratio is 1, and conversely;

i.e., if $$\frac{x-y}{y} \doteq 0, \text{ then } \frac{x}{y} \doteq 1.$$

For, since $$\frac{x-y}{y} = \frac{x}{y} - 1,$$

hence $$\frac{x}{y} - 1 \doteq 0, \text{ and } \frac{x}{y} \doteq 1. \qquad \text{[Art. 2, Cor. 2.}$$

Conversely, if $\frac{x}{y} \doteq 1$, then $\frac{x-y}{y} \doteq 0$.

For, by Art. 2, Cor. 1,

$$\frac{x}{y} - 1 \doteq 0; \; i.e., \; \frac{x-y}{y} \doteq 0.$$

5. Comparison of infinitesimals, and of infinites. Orders of magnitude. It has already been stated that any two variables are said to be of the same order of magnitude when the limit of their ratio is a finite number; that is to say, is neither infinite nor zero. In less precise language, two variables are of the same order of magnitude when one variable is neither infinitely larger nor infinitely smaller than the other. For instance, $k\beta$ is of the same order as β when k is any finite number; thus a finite multiplier or divisor does not affect the order of magnitude of any variable, whether infinitesimal, finite, or infinite.

In a problem involving infinitesimals, any one of them, α, may be chosen as a standard of comparison as to magnitude; then α is called the principal infinitesimal of the first order, and α^{-1} is called the principal infinite of the first order.

To test for the order n of any given infinitesimal β with reference to the principal infinitesimal α on which it depends, it is necessary to select an exponent n such that

$$\lim_{\alpha \doteq 0} \frac{\beta}{\alpha^n} = k,$$

wherein k is a finite constant, not zero.

When n is negative, β is infinite of order $-n$. An infinitesimal, or infinite of order zero, is a finite number.

E.g., to find the order of the variable $3x^4 - 4x^3$, with reference to x as the principal infinitesimal.

Comparing with x^2, x^3, x^4, in succession:

$$\lim_{x \doteq 0} \frac{3x^4 - 4x^3}{x^2} = \lim_{x \doteq 0} (3x^2 - 4x) = 0, \text{ not finite};$$

$$\lim_{x \doteq 0} \frac{3x^4 - 4x^3}{x^3} = \lim_{x \doteq 0} (3x - 4) = -4, \text{ finite};$$

$$\lim_{x \doteq 0} \frac{3x^4 - 4x^3}{x^4} = \lim_{x \doteq 0} \left(3 - \frac{4}{x}\right) = \infty, \text{ not finite};$$

hence $3x^4 - 4x^3$ is an infinitesimal of the same order of smallness as x^3; that is, of the third order.

The order of largeness of an infinite variable can be tested in a similar way. For instance, if x be taken as the principal infinite, let it be required to find the order of the variable $3x^4 - 4x^3$. Comparing with x^3 and x^4:

$$\lim_{x \doteq \infty} \frac{3x^4 - 4x^3}{x^3} = \lim_{x \doteq \infty} (3x - 4) = \infty;$$

$$\lim_{x \doteq \infty} \frac{3x^4 - 4x^3}{x^4} = \lim_{x \doteq \infty} \left(3 - \frac{4}{x}\right) = 3;$$

hence $3x^4 - 4x^3$ is an infinite of the same order of largeness as x^4, that is, of the fourth order.

The process of finding the limit of the ratio of two infinitesimals is facilitated by the following principle, based on theorem 8 of Art. 4: The limit of the quotient of two infinitesimals is not altered by adding to them (or subtracting from them) any two infinitesimals of higher order, respectively.

E.g.,
$$\lim_{x \doteq 0} \frac{3x^2 + x^4}{4x^2 - 2x^{\frac{5}{2}}} = \lim_{x \doteq 0} \frac{3x^2}{4x^2} = \frac{3}{4}.$$

From these definitions the following theorems are at once established:

Theorem 1. The product of two infinitesimals is another infinitesimal whose order is the sum of the orders of the factors.

Theorem 2. The quotient of an infinitesimal of order m by an infinitesimal of order n is an infinitesimal of order $m-n$.

Theorem 3. The order of an infinitesimal is not altered by adding or subtracting another infinitesimal of higher order.

6. Useful illustrations of infinitesimals of different orders.

Theorem 1. $\lim_{\theta \doteq 0} \frac{\sin \theta}{\theta} = 1$; $\lim_{\theta \doteq 0} \frac{\tan \theta}{\theta} = 1.$

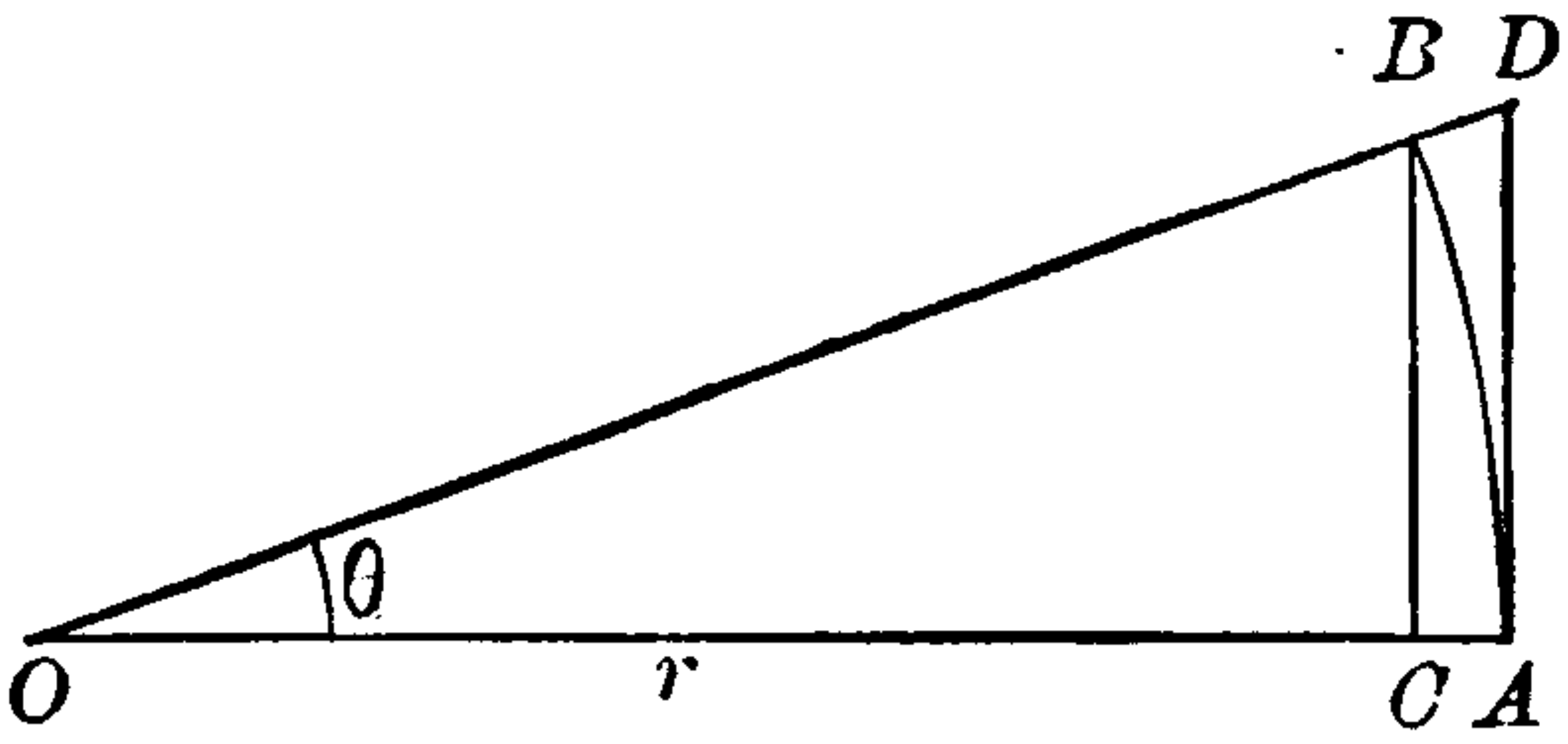

FIG. 1.

With O as a center and $OA = r$ as radius, describe the circular arc AB. Let the tangent at A meet OB produced in D; draw BC perpendicular to OA, cutting OA in C. Let the angle $AOB = \theta$ in radian measure, then

Radians = 57

$\theta = \frac{arc}{r}$

$$\text{arc } AB = r\theta,$$

$$CB < \text{arc } AB < AD, \quad \text{by geometry,}$$

i.e.,
$$r \sin \theta < r\theta < r \tan \theta,$$

$$\sin \theta < \theta < \tan \theta.$$

By dividing each member of these inequalities by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \sec\theta,$$

but $\sec\theta \doteq 1$, when $\theta \doteq 0$,

hence $$\lim_{\theta\doteq 0}\frac{\theta}{\sin\theta} = 1, \text{ and } \lim_{\theta\doteq 0}\frac{\sin\theta}{\theta} = 1.$$

Similarly, by dividing the inequalities by $\tan\theta$,

$$\cos\theta < \frac{\theta}{\tan\theta} < 1,$$

hence $$\lim_{\theta\doteq 0}\frac{\theta}{\tan\theta} = 1, \text{ and } \lim_{\theta\doteq 0}\frac{\tan\theta}{\theta} = 1.$$

Cor. 1. The numbers θ, $\sin\theta$, $\tan\theta$ are infinitesimals of the same order.

Cor. 2. The expressions $\sin\theta - \theta$, $\tan\theta - \theta$ are infinitesimal as to θ.

Theorem 2. If one angle θ, of a right triangle, be an infinitesimal of the first order, then the hypotenuse r and the adjacent side x are either both finite, or they are infinitesimals of the same order; and the opposite side y is an infinitesimal of order one higher than that of r and x.

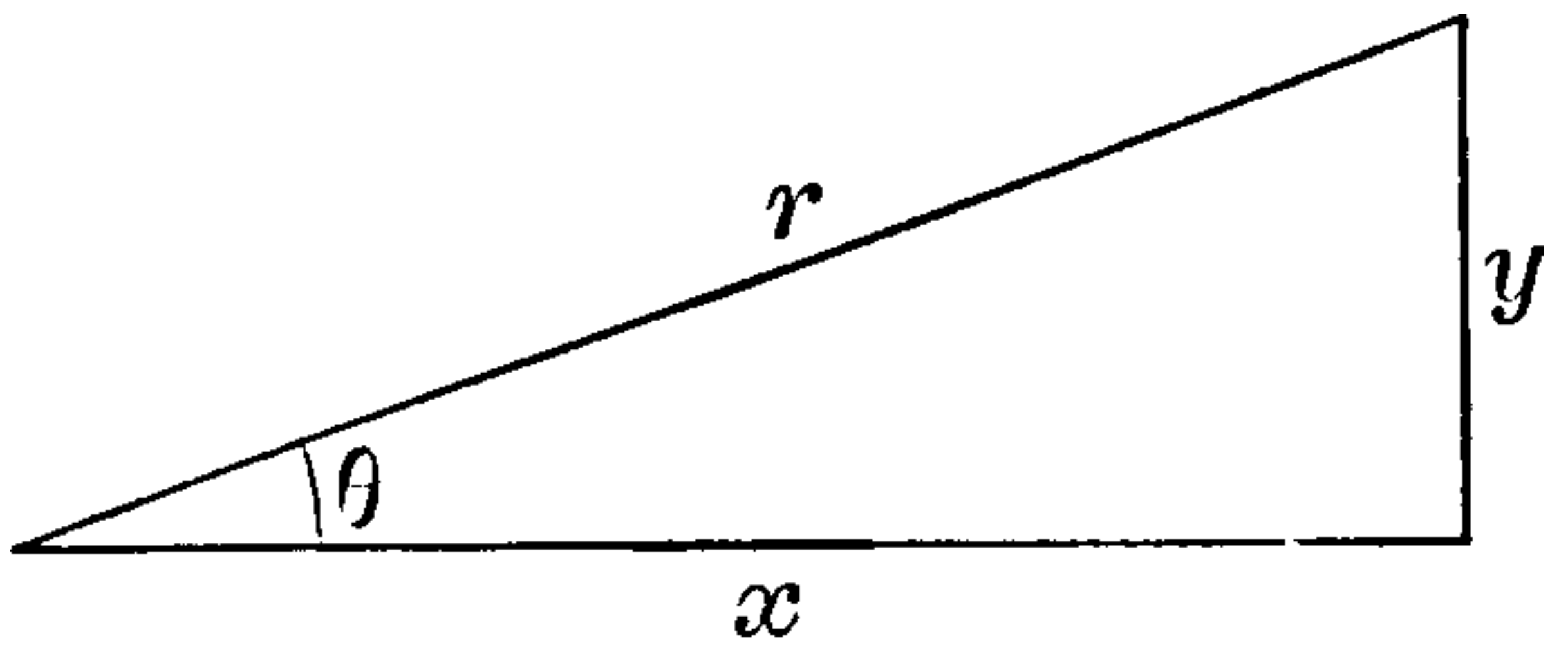

Fig. 2.

For $\frac{x}{r} = \cos\theta$, which approaches the value 1 as $\theta \doteq 0$; hence x, r are infinitesimals of the same order; which may be the order zero.

Also $$y = r\sin\theta,$$

and $\sin\theta$ is of order 1; therefore y is of order one higher than r, by theorem 1, Art. 5.

Cor. In the same case, if θ be of the first order, and if r and x be of the order n, then the difference between r and x is an infinitesimal of order $n+2$.

For $$r^2 - x^2 = y^2 = r^2 \sin^2\theta,\quad r - x = \frac{r^2 \sin^2\theta}{r+x};$$

but the orders of r^2, $\sin^2\theta$, $r+x$, are respectively $2n$, 2, n; therefore by theorem 2, Art. 5, $r-x$ is of order

$$2n + 2 - n = n + 2.$$

Theorem 3. The difference between the length of an infinitesimal arc of a circle and its chord is of at least the third order when the arc is the first order.

For, let CD be the arc, and CB, DB, tangents at its extremities. Then by elementary geometry

$$\text{chord } CD < \text{arc } CD < DB + BC.$$

Let the angle $BOD = \theta$ be taken as the principal infinitesimal. Then, since arc $CD = 2r\theta$, and r is finite, hence arc CD is of order **1**. Again, since AD is of order 1 (Th. 2), and angle $ADB = \theta$ is of order 1, hence DB is of order 1, and $DB - DA$ is of order 3 (Th. 2, Cor.); therefore $(DB + BC)$ − chord CD is of order 3.

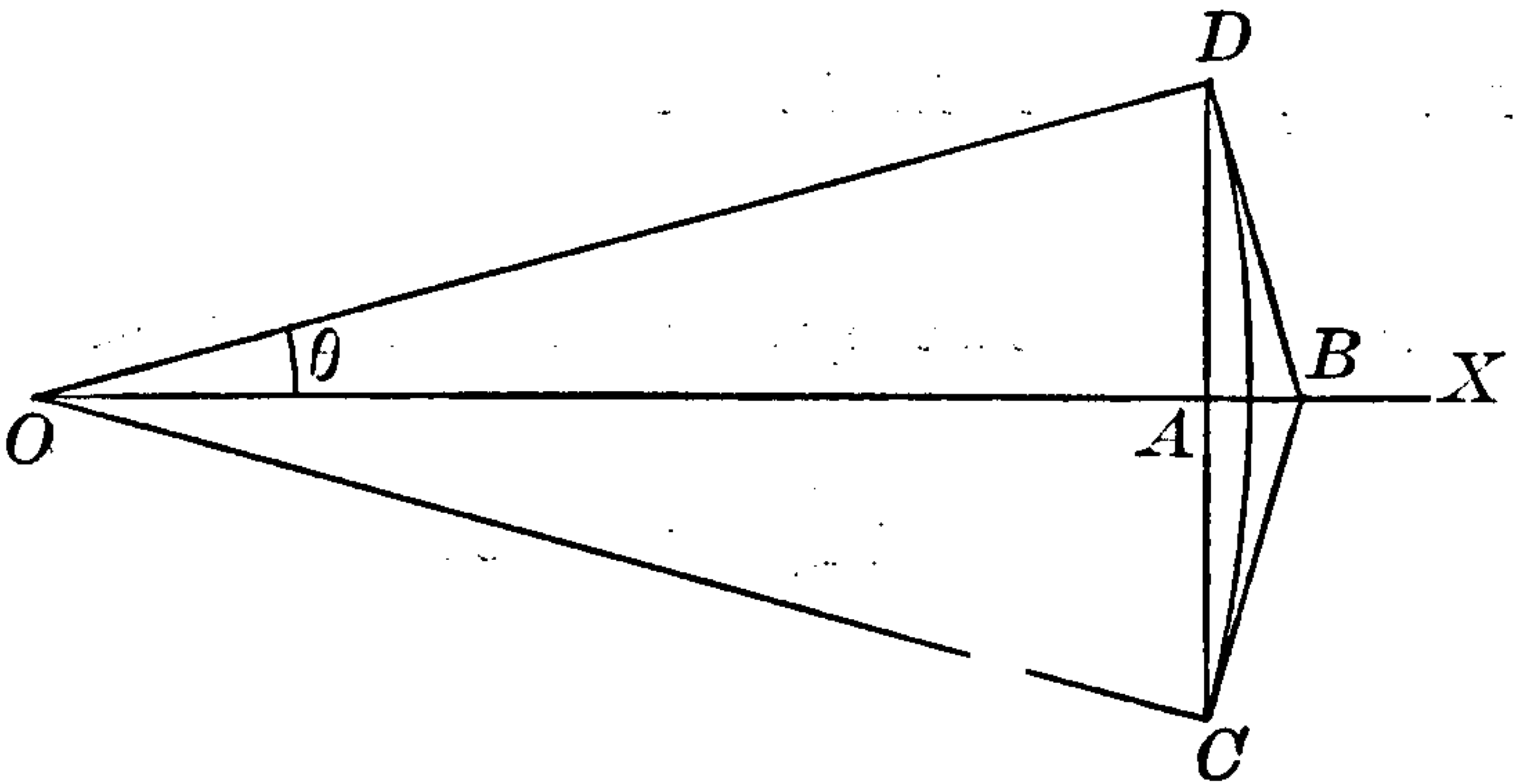

Fig. 3.

Hence arc CD − chord CD is of order, at least, three.

Theorem 4. The difference between the length of any infinitesimal arc (of finite curvature) and its chord, is an infinitesimal of, at least, the third order.

Note. The curvature is said to be finite when the limiting ratio of the length of a small chord to the acute angle between the tangents at its extremities is finite, and not zero.

If PQ be such an arc, the chord PQ and the angle TSP are, by hypothesis, infinitesimals of the same order.*

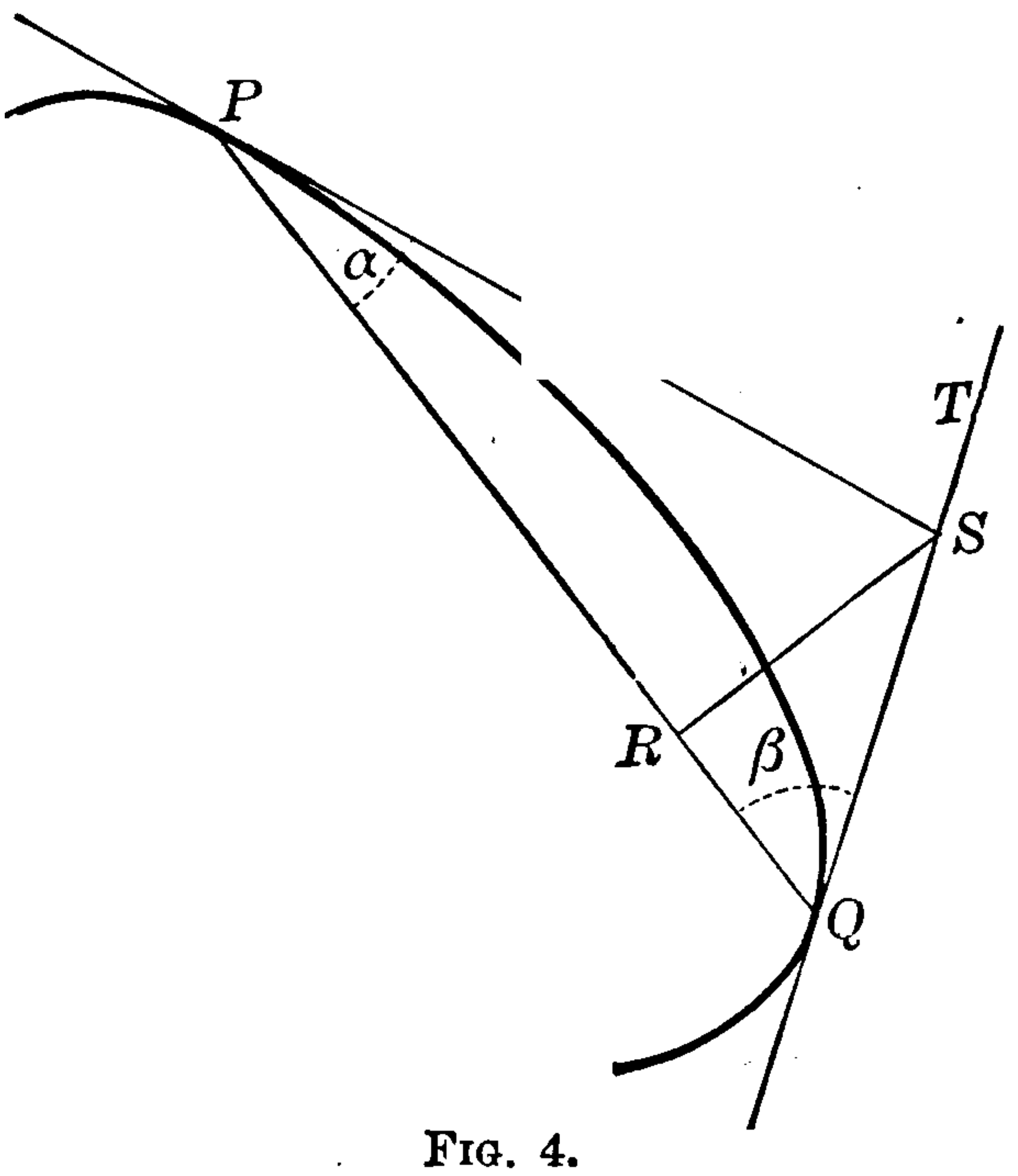

Fig. 4.

Let the angle TSP be the principal infinitesimal. Then, since

$$TSP = SQR + RPS,$$

it follows that the greater of the latter two angles, say SQR, is of the first order, while the other may be of the first or a higher order. Also, the greater of the two segments RQ, PR, say the latter, is of the first order, while RQ may be of the first or higher order.

Again, by theorem 2, QR, QS are of the same order, and PR, PS are of the same order.

Now $\quad \text{arc } QP - \text{chord } QP < QS + SP - QP, \quad$ [geom.

i.e., $\quad < (QS - QR) + (SP - RP);$

but since $\quad QS - QR = QS(1 - \cos\beta) = 2\,QS\sin^2\frac{\beta}{2},$

and, similarly, $\quad SP - RP = 2\,SP\sin^2\frac{\alpha}{2},$

and, since each of these products is, at least, of the third order, hence arc QP − chord QP is of, at least, the third order.

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* If TSP were of higher order than PQ, the curvature would be zero; if of lower order, the curvature would be infinite; the former is the case at an inflexion, the latter at a cusp.

EXERCISES

1. Let ABC be a triangle having a right angle at C; draw CD perpendicular to AB, DE perpendicular to CB, EF perpendicular to DB, FG perpendicular to EB; let the angle BAC be an infinitesimal of the first order, AB remaining finite. Prove that:

CD, CB are of order 1;
DB, DE are of order 2;
EB, EF, $(CB - CD)$ are of order 3;
FB, FG, $(DB - DE)$ are of order 4.

2. Of what order is the area of the triangle ABC? BCD? CDE?

3. A straight line, of constant length, slides between two rectangular straight lines, CAA', $CB'B$. Let AB, $A'B'$ be two positions of the line. Show that, in the limit, when the two positions coincide,

$$\frac{AA'}{BB'} = \frac{CB}{CA}.$$

7. Continuity of functions. When an independent variable x, in passing from a to b, passes through every intermediate value, it is called *continuous*.

A function $f(x)$ of an independent variable x is said to be continuous at any value x_1 when $f(x_1)$ is finite, real, and determinate, and such that in whatever way x approach x_1,

$$\lim_{x \doteq x_1} f(x) = f(x_1),$$

in which $f(x_1)$ is independent of the law of approach.

From the definition of a limit it follows that corresponding to a small increment of the variable the increment of the function is also small, and that corresponding to any number ϵ, previously assigned, another number δ can be determined, such that when h remains numerically less than δ the difference

$$f(x_1 \pm h) - f(x_1)$$

is numerically less than ϵ.

E.g., the function $f(x) = x^2 + 3x + 2$

is continuous at the value $x = 1$.

$$f(1) = 6,\quad f(1+h) = 6 + 5h + h^2,$$

$$f(1+h) - f(1) = 5h + h^2 = h(5+h).$$

If the difference $f(1+h) - f(1)$ is to be less than, say, $\frac{1}{1000000}$, it is only necessary that

$$h < \frac{1}{(5+h)1000000}.$$

If $\delta = \frac{1}{6000000}$, then for every value of h such that

$$h < \delta,$$

it is evident that $f(1+h) - f(1)$ is less than $\frac{1}{1000000}$.

When a function is continuous at every value of x within the interval from a to b, it is said to be continuous *within that interval.*

When a value x_1 exists at which any one of the preceding conditions is not fulfilled for a given function $\phi(x)$, the function is said to be *discontinuous* at $x = x_1$.

E.g., the function may become infinite, as $\frac{ax}{x-2}$, when $x \doteq 2$;

the function may be imaginary, as $\sqrt{9 - x^2}$, when $x^2 > 9$;

the function may be indeterminate, as $\sin\frac{1}{x}$, when $x \doteq 0$;

finally, the value of the function may depend upon the manner in which the variable approaches the value x_1, as in the function

$$f(x) = \frac{2 - 3^{\frac{1}{x}}}{1 - 3^{\frac{1}{x}}};$$

when $x = +h$, $f(x) \doteq 1$; when $x = -h$, $f(-h) \doteq 2$ as $h \doteq 0$.

A continuous function actually attains its limit for any value of the variable within the region of continuity, and the variable may be substituted directly.

It may be shown as on p. 14 that any polynomial

$$ax^n + bx^{n-1} + \cdots \qquad [n \text{ a positive integer.}$$

is continuous for every finite value of x.

The ordinary functions involving radicals and ratios are continuous only for certain intervals.

The trigonometric functions $\sin x$ and $\cos x$ are continuous for all real finite values of x; the other trigonometric functions are rationally expressible in terms of sine and cosine.

Show that $\tan x$ is discontinuous when $x = \frac{1}{2}\pi$.

The exponential function a^x and the logarithmic function $\log x$ are each continuous, the former for all finite values of x, the latter for all finite positive values of x [D. C., p. 31].

8. Comparison of simultaneous infinitesimal increments of two related variables. The last few articles were concerned with the principles to be used in comparing any two infinitesimals. In the illustrations given, the law by which each variable approached zero was assigned, or else the two variables were connected by a fixed relation; and the object was to find the limit of their ratio. The value of this limit gave the relative importance of the infinitesimals.

In the present article the particular infinitesimals compared are not the principal variables x, y themselves, but simultaneous increments h, k of these variables, as they start out from given values x_1, y_1 and vary in an assigned manner, as in the familiar instance of the abscissa and ordinate of a given curve.

The variables x, y are then to be replaced by their equivalents $x_1 + h$, $y_1 + k$, in which the increments h, k are themselves variables, and can, if desired, be both made to approach zero as a limit; for since y is supposed to be a continuous function of x, its increment can be made as small as desired by taking the increment of x sufficiently small.

The determination of the limit of the ratio of k to h, as h approaches zero, subject to an assigned relation between x and y, is the fundamental problem of the Differential Calculus.

E.g., let the relation be

$$y = x^2;$$

let x_1, y_1 be simultaneous values of the variables x, y; and when x changes to the value $x_1 + h$, let y change to the value $y_1 + k$. Then

$$y_1 = x_1^2,$$

$$y_1 + k = (x_1 + h)^2 = x_1^2 + 2x_1h + h^2;$$

hence $$k = 2x_1h + h^2.$$

This is a relation connecting the increments h, k.

Here it is to be observed that the relation between the infinitesimals h, k is not directly given, but has first to be derived from the known relation between x and y.

Let it next be required to compare these simultaneous increments by finding the limit of their ratio when they approach the limit zero.

By division,

$$\frac{k}{h} = 2x_1 + h;$$

hence, $$\underset{h\mathrm{l}\doteq 0}{\mathrm{im}} \frac{k}{h} = 2x_1.$$

This result may be expressed in familiar language by saying that when x increases through the value x_1, then y increases $2x_1$ times as much as x; and thus when x continues to increase uniformly, y increases more and more rapidly. For instance, when x passes through the value 4, and y through the value 16, the limit of the ratio of their increments is 8, and hence y is changing 8 times as fast as x; but when x is passing through 5, and y through 25, the limit of the ratio of their increments is 10, and y is changing 10 times as fast as x.

The following table will numerically illustrate the fact that the ratio of the infinitesimal increments h, k approaches nearer and nearer to some definite limit when h and k both approach the limit zero.

Let x_1, the initial value of x, be 4. Then y_1, the initial value of y, is 16. Let h, the increment of x, be 1. Then k, the corresponding increment of y, is found from

$$16 + k = (4 + h)^2;$$

thus $k = 9$, and $\frac{k}{h} = 9$. Next let h be successively diminished to the values .8, .6, .4, ... Then the corresponding values of k and of $\frac{k}{h}$ are as shown in the table:

$x = 4 + h$	$y = 16 + k$	k	$\frac{k}{h}$
$4 + 1$	25	9	9
$4 + .8$	23.04	7.04	8.8
$4 + .6$	21.16	5.16	8.6
$4 + .4$	19.36	3.36	8.4
$4 + .2$	17.64	1.64	8.2
$4 + .1$	16.81	.81	8.1
$4 + .01$	16.0801	.0801	8.01
. . .	. . .	. . .	. . .
$4 + h$	$16 + 8h + h^2$	$8h + h^2$	$8 + h$

Thus the ratio of corresponding increments takes the successive values 8.8, 8.6, 8.4, 8.2, 8.1, 8.01, ⋯, and can be brought as near to 8 as desired by taking h small enough.

As another example, let the relation between x and y be

$$y^2 = x^3.$$

Then
$$y_1^2 = x_1^3,$$
$$(y_1 + k)^2 = (x_1 + h)^3;$$

hence, by expansion and subtraction,

$$2\,y_1k + k^2 = 3\,x_1^2h + 3\,x_1h^2 + h^3,$$
$$k\,(2\,y_1 + k) = h\,(3\,x_1^2 + 3\,x_1h + h^2),$$
$$\frac{k}{h} = \frac{3\,x_1^2 + 3\,x_1h + h^2}{2\,y_1 + k}.$$

Therefore
$$\lim\frac{k}{h} = \lim\frac{3\,x_1^2 + 3\,x_1h + h^2}{2\,y_1 + k}, \text{ as } h \doteq 0,\ k \doteq 0,$$

and, by Art. 4, theorem 8,

$$\lim\frac{k}{h} = \frac{3\,x_1^2}{2\,y_1}.$$

The "initial values" of x, y, have been written with subscripts to show that only the increments h, k vary during the algebraic process, and also to emphasize the fact that the limit of the ratio of the simultaneous increments depends on the particular values through which the variables are passing, when they are supposed to take these increments. With this understanding the subscripts will hereafter be omitted. Moreover, the increments h, k will, for greater distinctness, be denoted by the symbols Δx, Δy, read "increment of x," "increment of y."

Ex. 1. If $x^2 + y^2 = a^2$, find $\lim\frac{\Delta y}{\Delta x}$. Let the initial values of the variables be denoted by x, y, and let the variables take the respective increments Δx, Δy, so that their new values $x + \Delta x$, $y + \Delta y$ shall still satisfy the given relation. Then

$$(x + \Delta x)^2 + (y + \Delta y)^2 = a^2.$$

By expansion, and subtraction,

$$2x\cdot\Delta x+(\Delta x)^2+2y\cdot\Delta y+(\Delta y)^2=0,$$

hence $$\Delta x(2x+\Delta x)=-\Delta y(2y+\Delta y),$$

and $$\frac{\Delta y}{\Delta x}=-\frac{2x+\Delta x}{2y+\Delta y}.$$

Therefore $$\lim_{\Delta x\doteq 0}\frac{\Delta y}{\Delta x}=-\lim_{\Delta x\doteq 0}\frac{2x+\Delta x}{2y+\Delta y}=-\frac{x}{y}.$$

The negative sign indicates that when Δx and the ratio $x:y$ are positive, Δy is negative; that is, an increase in x produces a decrease in y. This may be illustrated geometrically by drawing the circle whose equation is $x^2+y^2=a^2$ (Fig. 5).

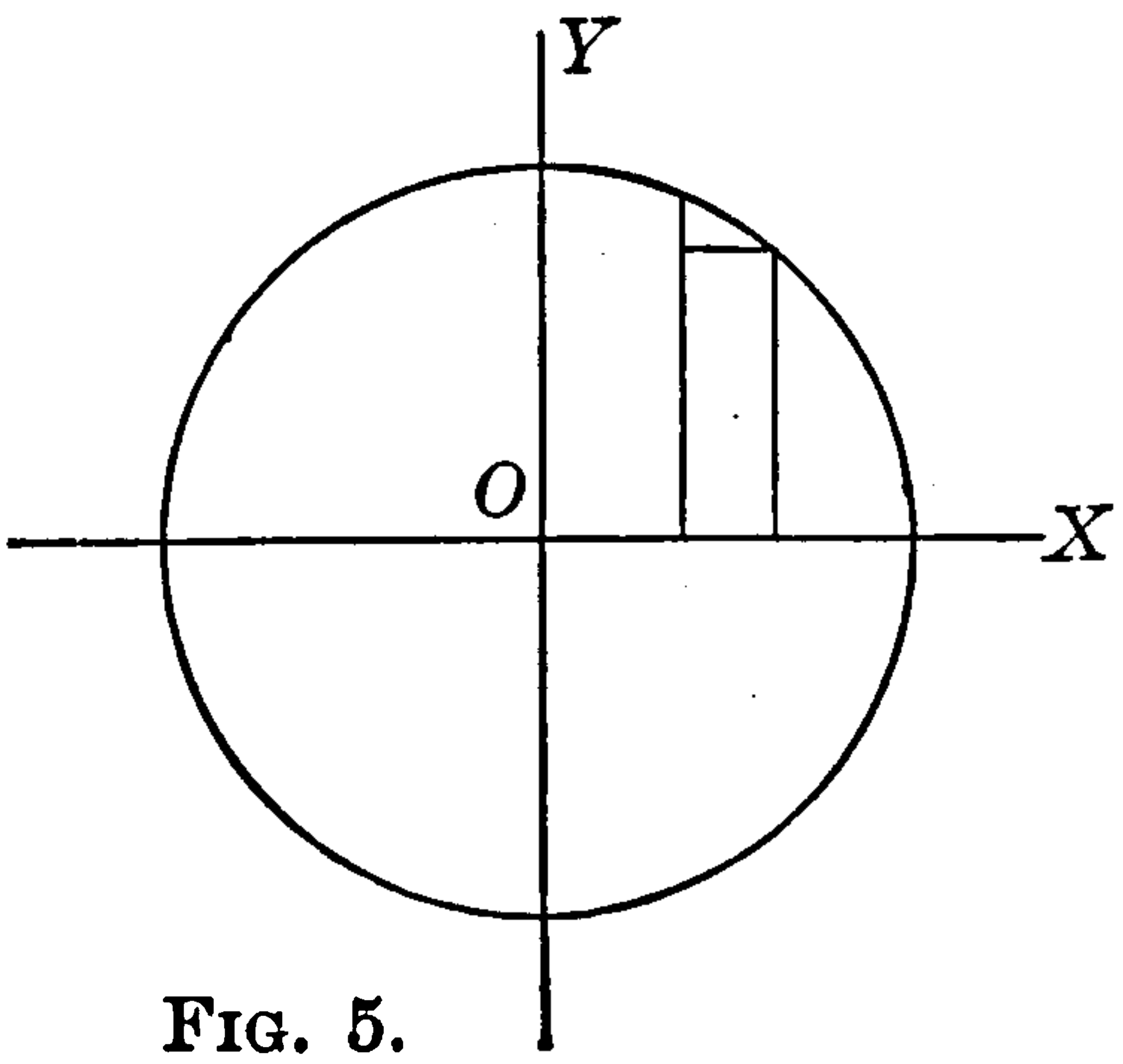

Fig. 5.

Ex. 2. If $x^2+y=y^2-2x$,

prove $$\lim_{\Delta x\doteq 0}\frac{\Delta y}{\Delta x}=\frac{2x+2}{2y-1}.$$

Similarly, when the relation between x and y is given in the explicit functional form

$$y=\phi(x),$$

then $$y+\Delta y=\phi(x+\Delta x),$$

and $$\Delta y=\phi(x+\Delta x)-\phi(x)=\Delta\phi(x),$$

hence $$\lim\frac{\Delta y}{\Delta x}=\lim\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}.$$

When the form of ϕ is given, the limit of this ratio can be evaluated, and expressed as a function of x. This function is then called the *derivative* of the function $\phi(x)$ with regard to the independent variable x.

The formal definition of the derivative of a function with regard to its variable is given in the next article.

9. Definition of a derivative. If to a variable a small increment be given, and if the corresponding increment of a continuous function of the variable be determined, then the limit of the ratio of the increment of the function to the increment of the variable, when the latter increment approaches the limit zero, is called the derivative of the function as to the variable.

If $\phi(x)$ be a finite and continuous function of x, and Δx a small increment given to x, then the derivative of $\phi(x)$ as to x is

$$\lim_{\Delta x \doteq 0} \left\{ \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} \right\} \equiv \lim_{\Delta x \doteq 0} \frac{\Delta\phi(x)}{\Delta x}$$

It is important to distinguish between $\lim \dfrac{\Delta\phi(x)}{\Delta x}$ and $\dfrac{\lim \Delta\phi(x)}{\lim \Delta x}$; that is, between the limit of the ratio of two infinitesimals and the ratio of their limits. The latter is indeterminate of the form $\dfrac{0}{0}$ and may have any value; but the former has usually a determinate value, as illustrated in the examples of the last article.

EXERCISES

1. Find the derivative of $x^2 - 2x$ as to x.

2. Find the derivative of $3x^2 - 4x + 3$ as to x.

3. Find the derivative of $\dfrac{1}{4x}$ as to x.

4. Find the derivative of $x^4 - 2 + \dfrac{3}{x^2}$ as to x.

10. Geometrical illustrations of a derivative. Some conception of the meaning and use of a derivative will be afforded by one or two geometrical illustrations.

Let $y = \phi(x)$ be a function of x that remains finite and continuous for all values of x between certain assigned constants a and b; and let the variables x, y be taken as the rectangular coördinates of a moving point. Then the relation between x and y is represented graphically, within the assigned bounds of continuity, by the curve whose equation is

$$y = \phi(x).$$

Let (x_1, y_1), (x_2, y_2) be the coördinates of two points P_1, P_2, on this curve. Then it is evident that the ratio

$$\frac{y_2 - y_1}{x_2 - x_1}$$

is equal to $\tan \alpha$, wherein α is the inclination angle of the secant line P_1P_2 to the x-axis. Let P_2 be moved nearer and nearer to coincidence with P_1, so that $x_2 \doteq x_1$, $y_2 \doteq y_1$. Then the secant line P_1P_2 approaches nearer and nearer to coincidence with the tangent line drawn at the point P_1, and the inclination-angle α of the secant approaches as a limit the inclination angle ϕ of the tangent line.

Hence,

$$\tan \alpha \doteq \tan \phi.$$

Thus $\dfrac{y_2 - y_1}{x_2 - x_1} \doteq \tan \phi$,

when $x_2 \doteq x_1$, $y_2 \doteq y_1$.

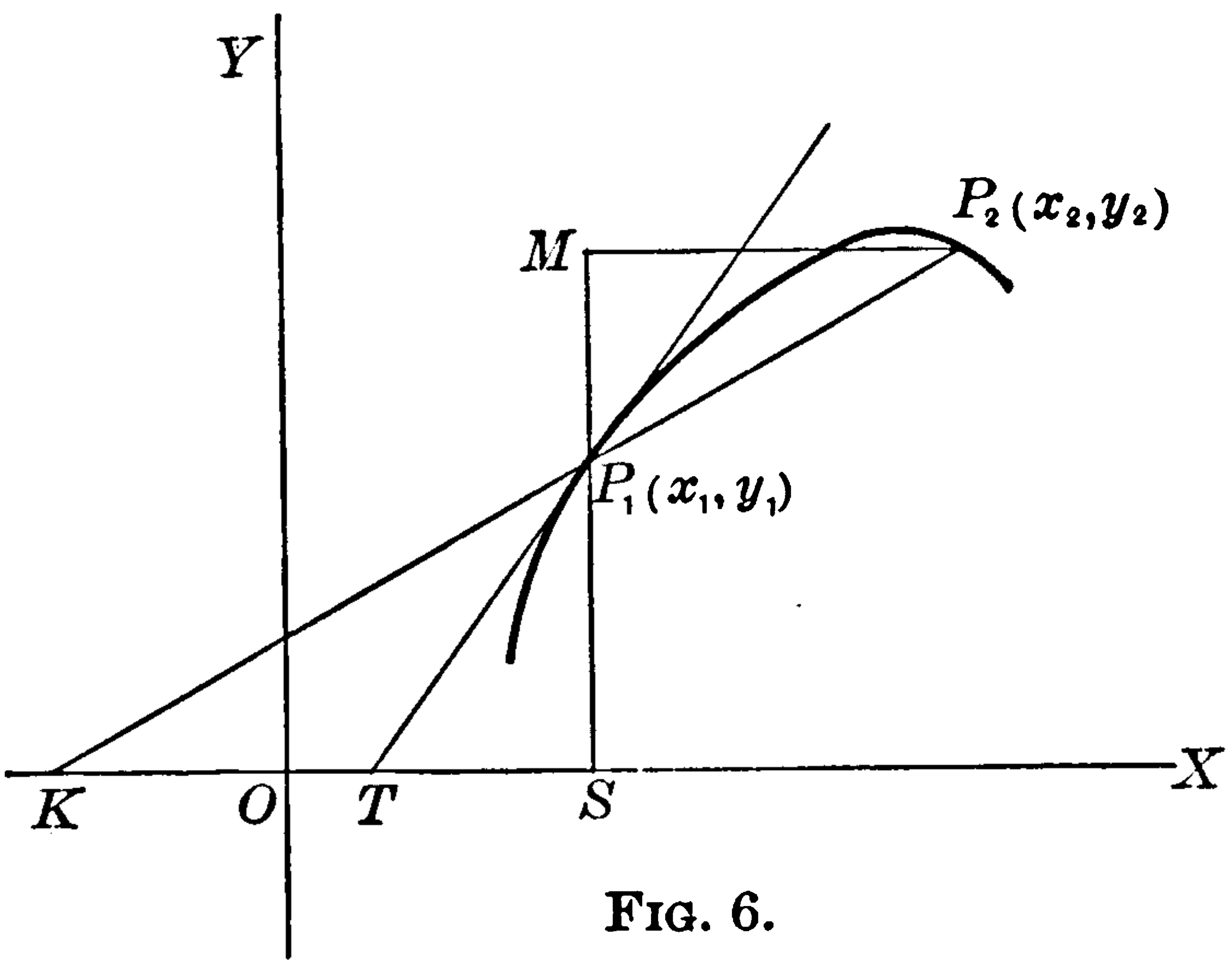

Fig. 6.

It may be observed that if x_2 be put directly equal to x_1, and y_2 to y_1, the ratio on the left would, in general, assume the indeterminate form $\frac{0}{0}$, as in other cases of finding the limit of the ratio of two infinitesimals; but it has just been shown that the ratio of the infinitesimals $y_2 - y_1$, $x_2 - x_1$ has, nevertheless, a determinate limit, viz., $\tan \phi$.

They are thus infinitesimals of the same order except when ϕ is 0 or $\frac{\pi}{2}$.

If the differences $x_2 - x_1$, $y_2 - y_1$ be denoted by Δx, Δy,

then $$x_2 = x_1 + \Delta x,\ y_2 = y_1 + \Delta y;$$

but, since $$y = \phi(x),$$

it follows that $$y_1 = \phi(x_1),\ y_2 = \phi(x_2);$$

hence the ratio of the simultaneous increments may be written in the various forms

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} = \frac{\phi(x_1 + \Delta x) - \phi(x_1)}{\Delta x}.$$

In the last form x is regarded as the independent variable and Δx as its independent increment; the numerator is the increment of the function $\phi(x)$, caused by the change of x from the value x_1 to the value $x_1 + \Delta x$. The limit of this ratio, as $\Delta x \doteq 0$, is the value of the derivative of the function $\phi(x)$ when x has the value x_1. Here x_1 stands for any assigned value of x. Thus the derivative of any continuous function $\phi(x)$ is another function of x which measures the slope of the tangent to the curve $y = \phi(x)$, drawn at the point whose abscissa is x.

Ex. Find the slope of the tangent line to the curve $y = \frac{2}{x^2}$ at the point (1, 2).

Here $$\tan\phi = \lim_{\Delta x \doteq 0} \frac{\dfrac{2}{(x + \Delta x)^2} - \dfrac{2}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \doteq 0} \frac{-2(2x + \Delta x)}{x^2(x + \Delta x)^2} = -\frac{4}{x^3}.$$

Hence $\tan \phi = -4$, when $x = 1$; and the equation of the tangent line at the point (1, 2) is $y - 2 = -4(x - 1)$.

As another illustration, if the coördinates of P be (x, y), and those of Q, $(x + \Delta x, y + \Delta y)$, then $MN = PR = \Delta x$, and $PS = RQ = \Delta y$. If the area $OAPM$ be denoted by z, then z is evidently some function of the abscissa x; also if area $OAQN$ be denoted by $z + \Delta z$, then the area $MNQP$ is Δz; it is the increment taken by the function z, when x takes the increment Δx. But $MNQP$ lies between the rectangles MR, MQ; hence

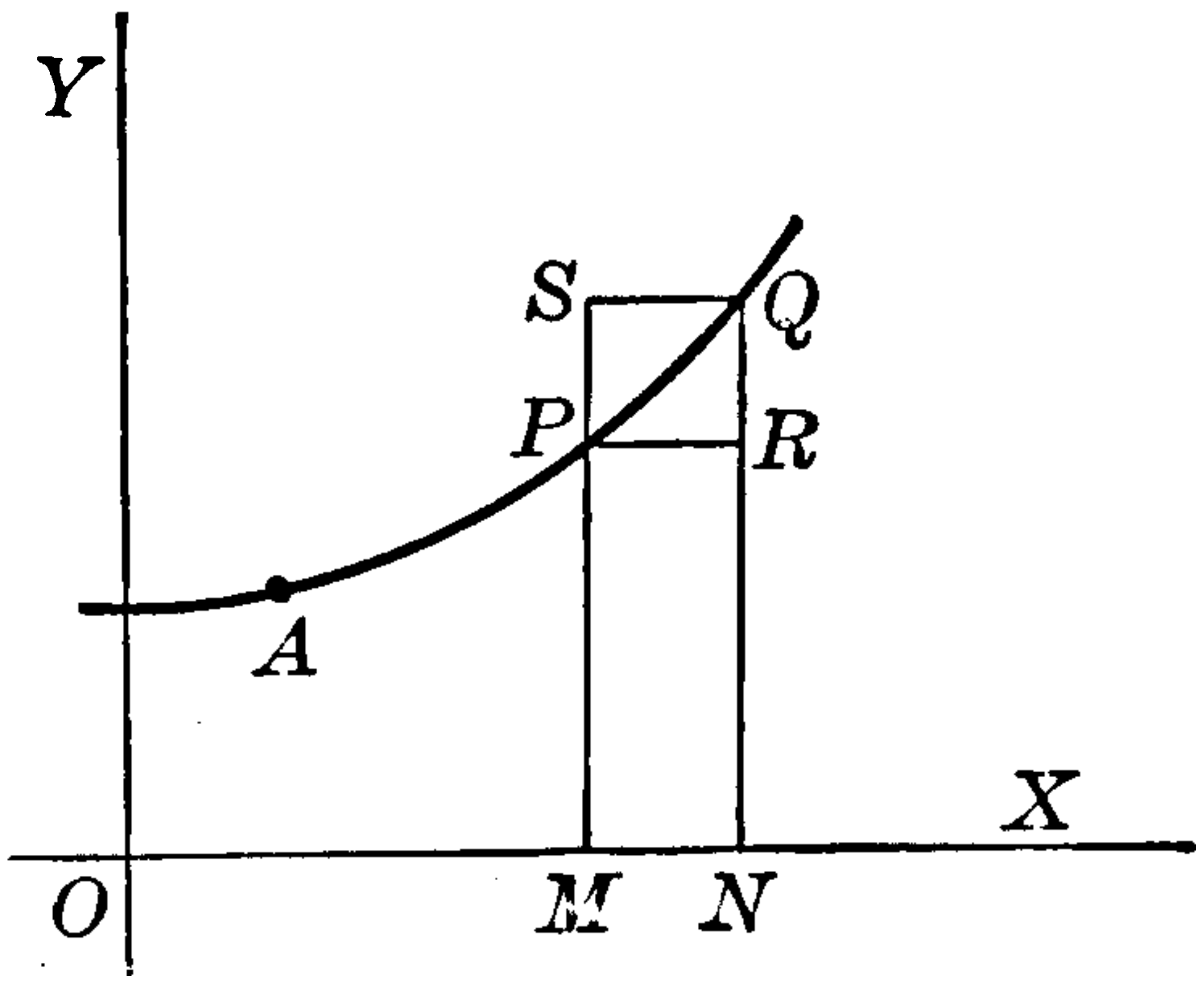

Fig. 7.

$$y\Delta x < \Delta z < (y + \Delta y)\Delta x,$$

and $$y < \frac{\Delta z}{\Delta x} < y + \Delta y.$$

Therefore, when Δx, Δy, Δz all approach zero,

$$\lim \frac{\Delta z}{\Delta x} = y.$$

Thus, if the ordinate and the area be each expressed as a function of the abscissa, the derivative of the area function with regard to the abscissa is equal to the ordinate function.

Ex. If the area included between a curve, the axis of x, and the ordinate whose abscissa is x, be given by the equation

$$z = x^3,$$

find the equation of the curve.

Here $$y = \lim \frac{\Delta z}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \doteq 0} [3x^2 + 3x\Delta x + (\Delta x)^2] = 3x^2.$$

11. The operation of differentiation. It has been seen in a number of examples that when the operation indicated by

$$\lim_{\Delta x \doteq 0} \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}$$

is performed on a given function $\phi(x)$, the result of the operation is another function of x. The latter function may have properties similar to those of $\phi(x)$, or it may be of an entirely different class.

The operation above indicated is for brevity denoted by the symbol $\frac{d\phi(x)}{dx}$, and the resulting derivative function by $\phi'(x)$; thus,

$$\frac{d\phi(x)}{dx} \equiv \lim_{\Delta x \doteq 0} \frac{A\phi(x)}{\Delta x} \equiv \lim_{\Delta^{x} \doteq 0} \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} \equiv \phi'(x).$$

The process of performing this indicated operation is called the *differentiation* of $\phi(x)$ with regard to x. The symbol* $\frac{d}{dx}$, when spoken of separately, is called the *differentiating operator,* and expresses that any function written after it is to be differentiated with regard to x, just as the symbol cos prefixed to $\phi(x)$ indicates that the latter is to have a certain operation performed upon it, namely, that of finding its cosine.

The process of differentiating $\phi(x)$ consists of the following steps:

1. Give a small increment to the variable.
2. Compute the resulting increment of the function.
3. Divide the increment of the function by the increment of the variable.
4. Obtain the limit of this quotient as the increment of the variable approaches zero.

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* This symbol is sometimes replaced by the single letter D.

EXERCISES

Find the derivatives of the following functions:

1. $5y^3 - 2y + 6$ as to y.
2. $7t^2 - 4t - 11t^3$ as to t.
3. $8u^3 - 4u + 10$ as to $2u$.
4. $2x^2 - 5x + 6$ as to $x - 3$.

This process will be applied in the next chapter to all the classes of functions whose continuity within certain intervals has been pointed out in Art. 7. It will be found that for each of them a derivative function exists; that is, that $\lim \dfrac{\Delta\phi(x)}{\Delta x}$ has a determinate and unique value, and that the curve $y = \phi(x)$ has a definite tangent within the range of continuity of the function.

A few curious functions have been devised, which are continuous and yet possess no definite derivative; but they do not present themselves in any of the ordinary applications of the Calculus. Again, there are a few functions for which $\lim \dfrac{\Delta\phi(x)}{\Delta x}$ has a certain value when $\Delta x \doteq 0$ from the positive side, and a different value when $\Delta x \doteq 0$ from the negative side; the derivative is then said to be *non-unique.*

Functions that possess a unique derivative within an assigned interval are said to be *differentiable* in that interval.

Ex. Show that the four steps of p. 24 do not apply at a discontinuity.

12. Increasing and decreasing functions. A good example of the use of the derivative is its application to finding the intervals of increasing or decreasing for a given function.

A function is called an *increasing* function if it increases as the variable increases and decreases as the variable decreases. A function is called a *decreasing* function if it decreases as the variable increases, and increases as the variable decreases.

E.g., the function $x^2 + 4$ decreases as x increases from $-\infty$ to 0, but it increases as x increases from 0 to $+\infty$. Thus $x^2 + 4$ is a decreasing function while x is negative, and an increasing function while x is positive. This is well shown by the locus of the equation $y=x^2+4$ (Fig. 8).

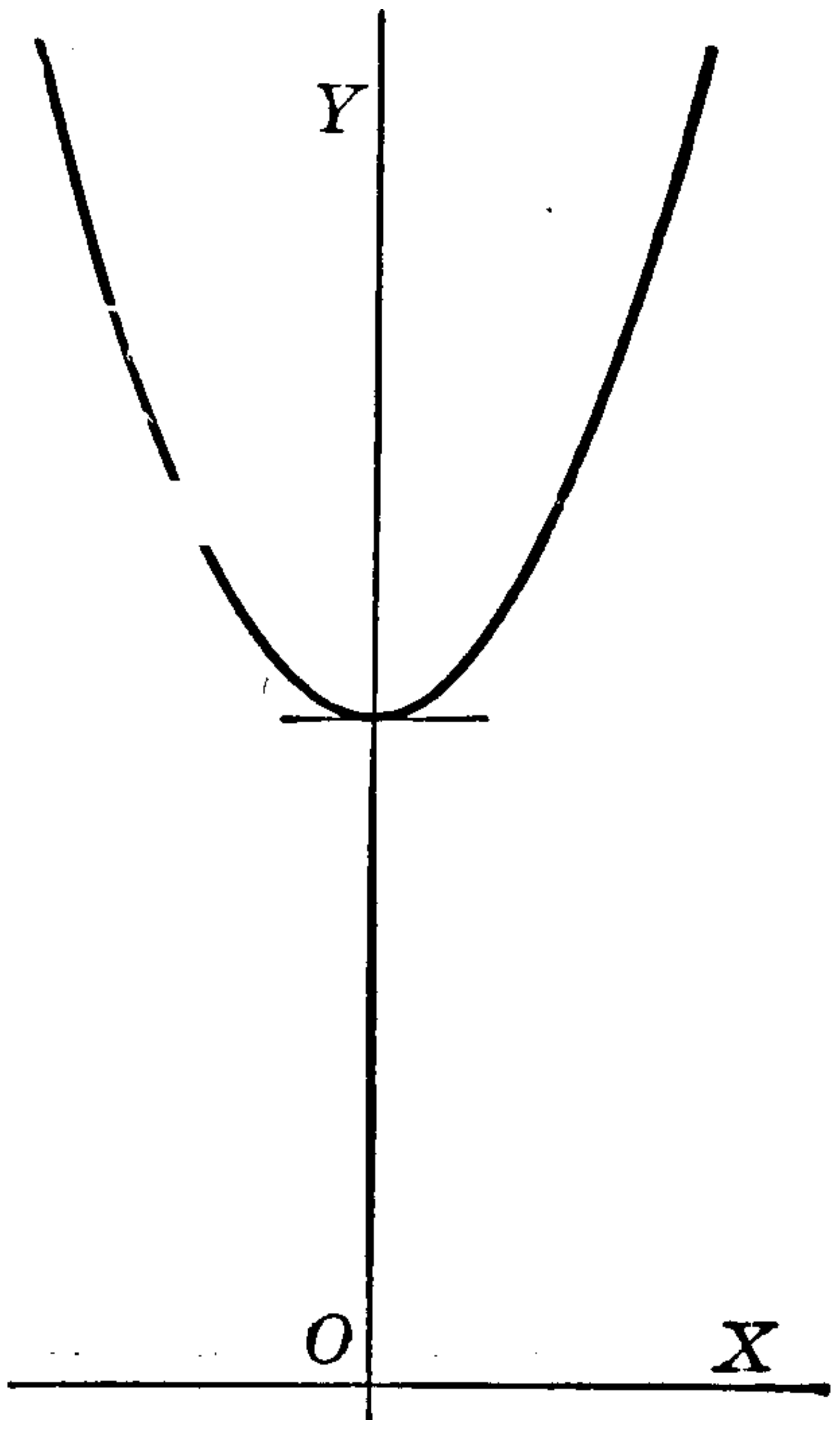

FIG. 8.

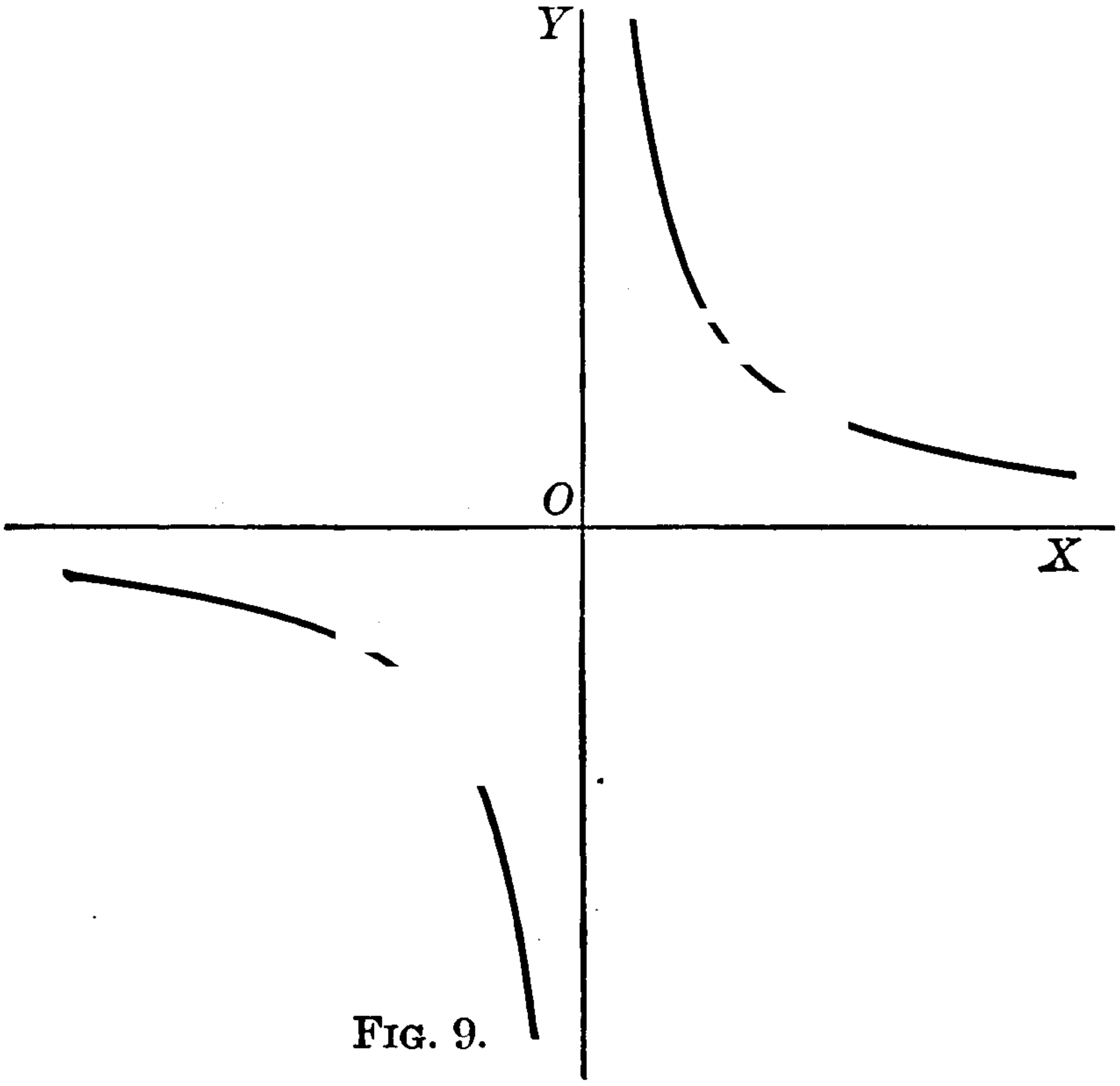

FIG. 9.

Again, the form of the curve $y=\frac{1}{x}$ shows that $\frac{1}{x}$ is a decreasing function, as x passes from $-\infty$ to 0, and also a decreasing function, as x passes from 0 to $+\infty$. When x passes through 0, the function changes discontinuously from the value $-\infty$ to the value $+\infty$ (Fig. 9).

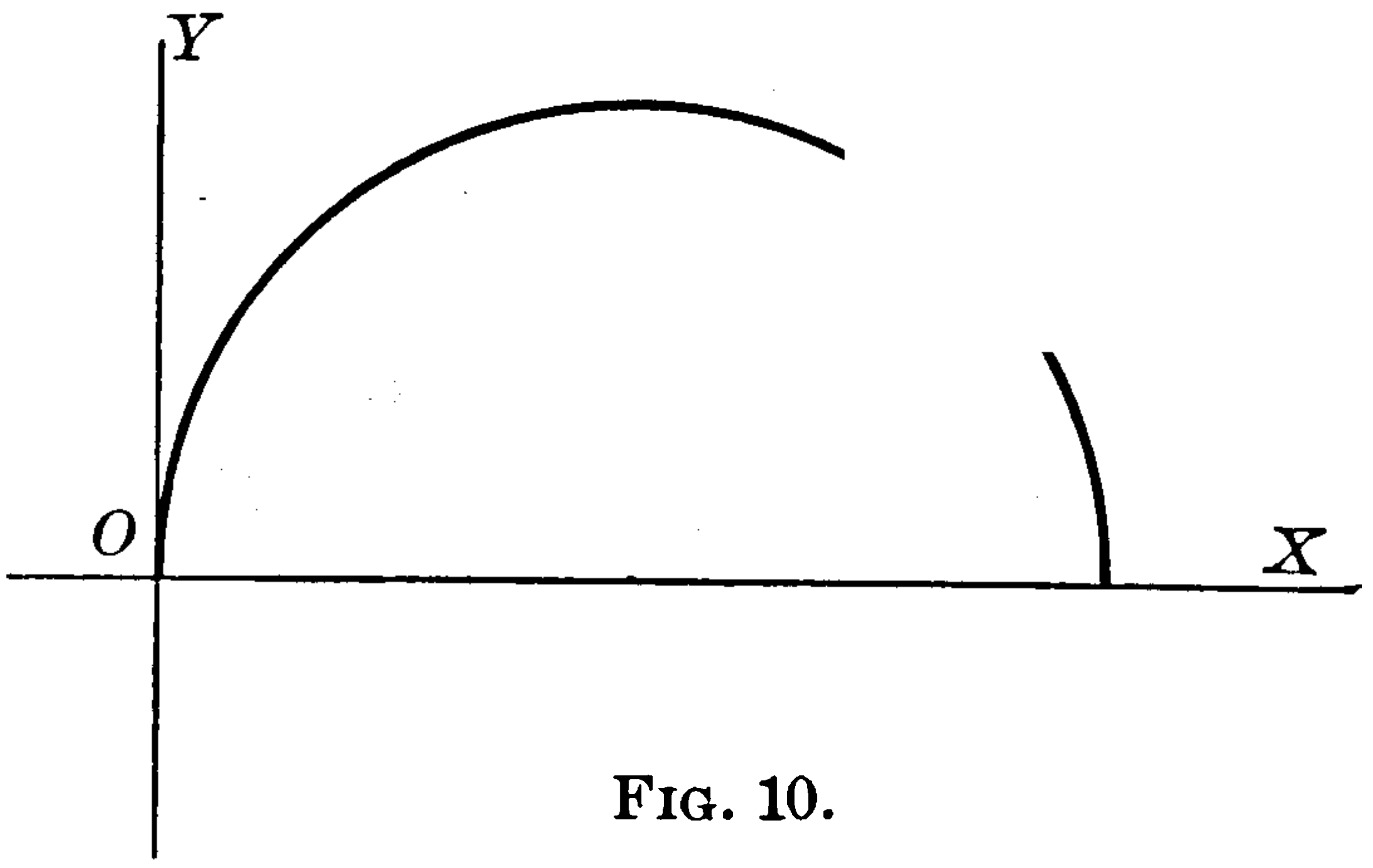

FIG. 10.

Most functions are increasing functions for some values of the variable, and decreasing functions for others.

E.g., $\overset{+}{\sqrt{2rx-x^2}}$ is an increasing function from $x=0$ to $x=r$, and a decreasing function from $x=r$ to $x=2r$ (Fig. 10).

A function is said to be an increasing or decreasing function *in the vicinity* of a given value of x according as it increases or decreases as x increases through a small interval including this value.

13. Algebraic test of the intervals of increasing and decreasing. Let $y = \phi(x)$ be a function of x, and let it be real, continuous, and differentiable for all values of x from a to b. Then by definition y is increasing or decreasing at a point $x = x_1$, according as

$$\phi(x_1 + \Delta x) - \phi(x_1)$$

is positive or negative, where Δx is a small positive number.

The sign of this expression is not changed if it be divided by Δx, no matter how small Δx may be; hence $\phi(x)$ is an increasing or a decreasing function at the value x_1, according as

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \left\{ \frac{\phi(x_1 + \Delta x) - \phi(x_1)}{\Delta x} \right\} = \phi'(x_1)$$

is positive or negative.

Thus the intervals in which $\phi(x)$ is an increasing function are the same as the intervals in which $\phi'(x)$ is positive.

Ex. Find the intervals in which the function

$$\phi(x) = 2x^3 - 9x^2 + 12x - 6$$

is increasing or decreasing. The derivative is

$$\phi'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2);$$

hence, as x passes from $-\infty$ to 1, the derived function $\phi'(x)$, is positive and $\phi(x)$ increases from $\phi(-\infty)$ to $\phi(1)$, *i.e.*, from $\phi = -\infty$ to $\phi = -1$; as x passes from 1 to 2, $\phi'(x)$ is negative, and $\phi(x)$ decreases from $\phi(1)$ to $\phi(2)$, *i.e.*, from -1 to -2; and as x passes from 2 to $+\infty$, $\phi'(x)$ is positive, and $\phi(x)$ increases from $\phi(2)$ to $\phi(\infty)$, *i.e.*, from -2 to $+\infty$. The locus of the equation $y = \phi(x)$ is shown in Fig. 11. At points where $\phi'(x) = 0$, the function $\phi(x)$ is neither increasing nor decreasing. At such points the tangent is parallel to the axis of x. Thus in this illustration, at $x = 1$, $x = 2$, the tangent is parallel to the x-axis.

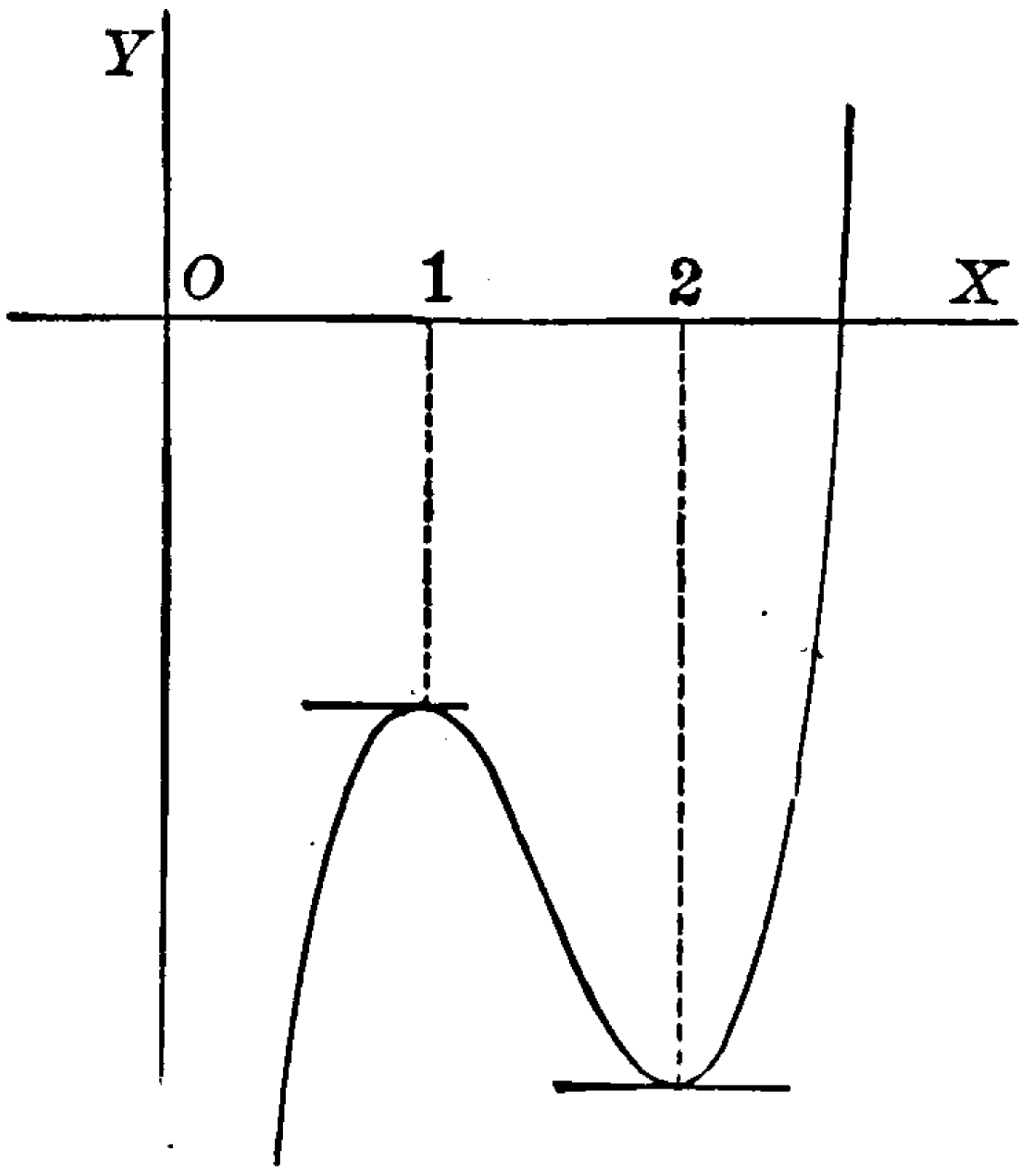

FIG. 11.

EXERCISES

1. Find the intervals of increasing and decreasing for the function

$$\phi(x) \equiv x^3 + 2x^2 + x - 4.$$

Here $\phi'(x) = 3x^2 + 4x + 1 = (3x + 1)(x + 1).$

The function increases from $x = -\infty$ to $x = -1$; decreases from $x = -1$ to $x = -\frac{1}{3}$; increases from $x = -\frac{1}{3}$ to $x = \infty$.

2. Find the intervals of increasing and decreasing for the function

$$y = x^3 - 2x^2 + x - 4,$$

and show where the curve is parallel to the x-axis.

3. At how many points can the slope of the tangent to the curve

$$y = 2x^3 - 3x^2 + 1$$

be 1? -1? Find the points.

4. Compute the angle at which the following curves intersect:

$$y = 3x^2 - 1, \quad y = 2x^2 + 3.$$

14. Differentiation of a function of a function. Suppose that y, instead of being given directly as a function of x, is expressed as a function of another variable u, which is itself expressed as a function of x. Let it be required to find the derivative of y with regard to the independent variable x.

Let $y = f(u)$, in which u is a function of x. When x changes to the value $x + \Delta x$, let u and y, under the given relations, change to the values $u + \Delta u$, $y + \Delta y$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{\Delta x};$$

hence, equating limits,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}.$$

This result may be stated as follows:

The derivative of a function of u with regard to x is equal to the product of the derivative of the function with regard to u, and the derivative of u with regard to x.

EXERCISES

1. Given $y = 3u^2 - 1$, $u = 3x^2 + 4$; find $\frac{dy}{dx}$.

$$\frac{dy}{du} = 6u, \quad \frac{du}{dx} = 6x;$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 36ux = 36x(3x^2 + 4).$$

2. Given $y = 3u^2 - 4u + 5$, $u = 2x^3 - 5$; find $\frac{dy}{dx}$.

3. Given $y = \frac{1}{u}$, $u = 5x^2 - 2x + 4$; find $\frac{dy}{dx}$.

4. Given $y = 3u^2 + \frac{1}{3u^2}$, $u = \frac{x^3}{3} + \frac{3}{x^3}$; find $\frac{dy}{dx}$.

CHAPTER II

DIFFERENTIATION OF THE ELEMENTARY FORMS

In recent articles, the meaning of the symbol $\frac{dy}{dx}$ was explained and illustrated; and a method of expressing its value, as a function x, was exemplified, in cases in which y was a simple algebraic function of x, by direct use of the definition. This method is not always the most convenient one in the differentiation of more complicated functions.

The present chapter will be devoted to the establishment of some general rules of differentiation which will, in many cases, save the trouble of going back to the definition.

The next five articles treat of the differentiation of algebraic functions and of algebraic combinations of other differentiable functions.

15. Differentiation of the product of a constant and a variable.

Let $$y = cx.$$

Then $$y + \Delta y = c(x + \Delta x),$$

$$\Delta y = c(x + \Delta x) - cx = c\Delta x,$$

$$\frac{\Delta y}{\Delta x} = c;$$

therefore $$\frac{dy}{dx} = c.$$

Cor. If $y = cu$, where u is a function of x, then, by Art. 14,

$$\frac{d(cu)}{dx} = c\frac{du}{dx}. \tag{1}$$

The derivative of the product of a constant and a variable is equal to the constant multiplied by the derivative of the variable.

16. Differentiation of a sum.

Let
$$y = f(x) + \phi(x) + \psi(x).$$

Then
$$y + \Delta y = f(x + \Delta x) + \phi(x + \Delta x) + \psi(x + \Delta x),$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} + \frac{\psi(x + \Delta x) - \psi(x)}{\Delta x}.$$

Therefore, by equating the limits of both members,

$$\frac{dy}{dx} = f'(x) + \phi'(x) + \psi'(x). \quad \text{[Art. 3, Th. 5.}$$

Cor. 1. If $y = u + v + w$, in which u, v, w are functions of x, then

$$\frac{d}{dx}(u + v + w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}. \tag{2}$$

The derivative of the sum of a finite number of functions is equal to the sum of their derivatives.

Cor. 2. If $y = u + c$, c being a constant, then

$$y + \Delta y = u + \Delta u + c;$$

hence,
$$\Delta y = \Delta u,$$

and
$$\frac{dy}{dx} = \frac{du}{dx}.$$

The last equation asserts that all functions which differ from each other only by an additive constant have the same derivative.

Geometrically, the addition of a constant has the effect of moving the curve $y = u(x)$ parallel to the y axis; this operation will obviously not change the slope at points that have the same x.

From (2), $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dc}{dx};$$

but from the fourth equation above,

$$\frac{dy}{dx} = \frac{du}{dx};$$

hence, it follows that $$\frac{dc}{dx} = 0.$$

The derivative of a constant is zero.

If the number of functions be infinite, theorem 5 of Art. 3 may not apply; that is, the limit of the sum may not be equal to the sum of the limits, and hence the derivative of the sum may not be equal to the sum of the derivatives. Thus the derivative of an infinite series cannot always be found by differentiating it term by term.

17. Differentiation of a product.

Let $$y = f(x)\phi(x).$$

Then $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)\phi(x+\Delta x) - f(x)\phi(x)}{\Delta x}.$$

By subtracting and adding $f(x)\phi(x+\Delta x)$ in the numerator, this result may be rearranged thus:

$$\frac{\Delta y}{\Delta x} = \phi(x+\Delta x)\frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x)\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x}.$$

Now let Δx approach zero, using Art. 3, theorems 5, 6, and noting that the first factor $\phi(x+\Delta x)$ approaches $\phi(x)$ since by hypothesis $\phi(x)$ is continuous (Art. 7). Then

$$\frac{dy}{dx} = \phi(x)f'(x) + f(x)\phi'(x).$$

Cor. 1. By writing $u = \phi(x)$, $v = f(x)$, this result can be more concisely written,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3}$$

The derivative of the product of two functions is equal to the sum of the products of the first factor by the derivative of the second, and the second factor by the derivative of the first.

This rule for differentiating a product of two functions may be stated thus: Differentiate the product, treating the first factor as constant, then treating the second factor as constant, and add the two results.

Cor. 2. To find the derivative of the product of three functions uvw.

Let $$y = uvw.$$

By (3), $$\frac{dy}{dx} = w\frac{d}{dx}(uv) + uv\frac{dw}{dx}$$
$$= w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + uv\frac{dw}{dx}.$$

The result may be written in the form

$$\frac{d(uvw)}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}. \tag{4}$$

By application of the same process to the product of 4, 5, ⋯, n functions, the following rule is at once deduced:

The derivative of the product of any finite number of factors is equal to the sum of the products obtained by multiplying the derivative of each factor by all the other factors.

18. Differentiation of a quotient.

Let $$y = \frac{f(x)}{\phi(x)}.$$

Then $$y + \Delta y = \frac{f(x + \Delta x)}{\phi(x + \Delta x)},$$

$$\frac{\Delta y}{\Delta x}=\frac{\dfrac{f(x+\Delta x)}{\phi(x+\Delta x)}-\dfrac{f(x)}{\phi(x)}}{\Delta x}$$

$$=\frac{\phi(x)f(x+\Delta x)-f(x)\phi(x+\Delta x)}{\Delta x\,\phi(x)\phi(x+\Delta x)}.$$

By subtracting and adding $\phi(x)f(x)$ in the numerator, this expression may be written

$$\frac{\Delta y}{\Delta x}=\frac{\phi(x)\left\{\dfrac{f(x+\Delta x)-f(x)}{\Delta x}\right\}-f(x)\left\{\dfrac{\phi(x+\Delta x)-\phi(x)}{\Delta x}\right\}}{\phi(x)\phi(x+\Delta x)}.$$

Hence, by equating limits,

$$\frac{dy}{dx}=\frac{\phi(x)f'(x)-f(x)\phi'(x)}{[\phi(x)]^2}. \qquad \text{[Art. 3, Ths. 6, 7.}$$

This result may be written in the briefer form

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}. \qquad (5)$$

The derivative of a fraction, the quotient of two functions, is equal to the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, divided by the square of the denominator.

19. Differentiation of a commensurable power of a function.

Let $y=u^n$, in which u is a function of x. Then there are three cases to consider.

1. n a positive integer.
2. n a negative integer.
3. n a commensurable fraction.

1. n a positive integer.

This is a particular case of (4), the factors $u, v, w, \cdots$ all being equal. Thus

$$\frac{dy}{dx}=nu^{n-1}\frac{du}{dx}$$

2. n a negative integer.

Let $n = -m$, in which m is a positive integer.

Then $$y = u^n = u^{-m} = \frac{1}{u^m},$$

and $$\frac{dy}{dx} = \frac{-mu^{m-1}}{u^{2m}} \cdot \frac{du}{dx} \text{ by (5), and Case (1)}$$

$$= -mu^{-m-1}\frac{du}{dx};$$

hence $$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

3. n a commensurable fraction.

Let $n = \frac{p}{q}$, where p, q are both integers, which may be either positive or negative.

Then $$y = u^n = u^{\frac{p}{q}};$$

hence $$y^q = u^p,$$

and * $$\frac{d}{dx}(y^q) = \frac{d}{dx}(u^p);$$

i.e., $$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}.$$

Solving for the required derivative,

$$\frac{dy}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx};$$

hence $$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}. \tag{6}$$

The derivative of any commensurable power of a function is equal to the exponent of the power multiplied by the power with its exponent diminished by unity, multiplied by the derivative of the function.

---

* If two functions be identical, their derivatives are identical.

These theorems will be found sufficient for the differentiation of any function that involves only the operations of addition, subtraction, multiplication, division, and involution in which the exponent is an integer or commensurable fraction.

The following examples will serve to illustrate the theorems, and will show the combined application of the general forms **(1)** to **(6)**.

ILLUSTRATIVE EXAMPLES

1. $y = \frac{3x^2 - 2}{x + 1}$; find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(3x^2 - 2) - (3x^2 - 2)\frac{d}{dx}(x+1)}{(x+1)^2}. \qquad \text{by } \mathbf{(5)}$$

$$\frac{d}{dx}(3x^2 - 2) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(2) \qquad \text{by } \mathbf{(2)}$$

$$= 6x. \qquad \text{by } \mathbf{(1)}, \mathbf{(6)}$$

$$\frac{d}{dx}(x+1) = \frac{dx}{dx} = 1. \qquad \text{by } \mathbf{(2)}$$

Substitute these results in the expression for $\frac{dy}{dx}$. Then

$$\frac{dy}{dx} = \frac{(x+1)6x - (3x^2 - 2)}{(x+1)^2} = \frac{3x^2 + 6x + 2}{(x+1)^2}.$$

2. $u = (3s^2 + 2)\sqrt{1 + 5s^2}$; find $\frac{du}{ds}$.

$$\frac{du}{ds} = (3s^2 + 2)\frac{d}{ds}\sqrt{1 + 5s^2} + \sqrt{1 + 5s^2} \cdot \frac{d}{ds}(3s^2 + 2). \qquad \text{by } \mathbf{(3)}$$

$$\frac{d}{ds}\sqrt{1 + 5s^2} = \frac{d}{ds}(1 + 5s^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1 + 5s^2)^{-\frac{1}{2}}\frac{d}{ds}(1 + 5s^2) \qquad \text{by } \mathbf{(6)}$$

$$= \frac{5s}{\sqrt{1 + 5s^2}}.$$

$$\frac{d}{ds}(3s^2 + 2) = 6s. \qquad \text{by } \mathbf{(6)}$$

Substitute these values in the expression for $\frac{du}{ds}$. Then

$$\frac{du}{ds}=\frac{5\,s\,(3\,s^2+2)}{\sqrt{1+5\,s^2}}+6\,s\sqrt{1+5\,s^2}=\frac{45\,s^3+16\,s}{\sqrt{1+5\,s^2}}.$$

3. $y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$; find $\frac{dy}{dx}$.

First, as a quotient,

$$\frac{dy}{dx}=\frac{(\sqrt{1+x^2}-\sqrt{1-x^2})\frac{d}{dx}(\sqrt{1+x^2}+\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})^2}$$

$$-\frac{(\sqrt{1+x^2}+\sqrt{1-x^2})\frac{d}{dx}(\sqrt{1+x^2}-\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})^2} \qquad \text{by (5)}$$

$$\frac{d}{dx}(\sqrt{1+x^2}+\sqrt{1-x^2})=\frac{d}{dx}\sqrt{1+x^2}+\frac{d}{dx}\sqrt{1-x^2}. \qquad \text{by (2)}$$

$$\frac{d}{dx}\sqrt{1+x^2}=\frac{d}{dx}(1+x^2)^{\frac{1}{2}}=\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\frac{d}{dx}(1+x^2). \qquad \text{by (6)}$$

$$\frac{d}{dx}(1+x^2)=2\,x. \qquad \text{by (2) and (6)}$$

Similarly for the other terms. Combining the results,

$$\frac{dy}{dx}=\frac{-2}{x^3}\left(1+\frac{1}{\sqrt{1-x^4}}\right).$$

Ex. 3 may also be worked by first rationalizing the denominator.

EXERCISES

Find the x-derivatives of the following functions:

1. $y=x^{10}$.

2. $y=x^{-8}$.

3. $y=c\sqrt{x}$.

4. $y=\frac{2}{\sqrt{x}}-\frac{\sqrt[3]{x}}{3}$.

5. $y=\sqrt[4]{x^{-5}}$.

6. $y=(x+a)^n$.

7. $y=x^n+a^n$.

8. $y=\frac{x}{\sqrt{a^2-x^2}}$.

9. $y=\frac{x+3}{x^2+2}$.

10. $y=(x+1)\sqrt{x+2}$.

11. $y=\frac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}$.

12. $y=\sqrt{\frac{1+x}{1-x}}$.

13. $y = \dfrac{x}{x + \sqrt{1 - x^2}}.$

14. $y = (2a^{\frac{1}{2}} + x^{\frac{1}{2}})\sqrt{a^{\frac{1}{2}} + x^{\frac{1}{2}}}.$

15. $y = \left\{\dfrac{x}{1 + \sqrt{1 - x^2}}\right\}^n.$

16. $y = \sqrt{\dfrac{1 - x^2}{(1 + x^2)^3}}.$

17. $y = \dfrac{x^n + 1}{x^n - 1}.$

18. $y = \dfrac{1}{(a + x)^m} \cdot \dfrac{1}{(b + x)^n}.$

19. $y = \dfrac{3x^3 + 2}{x(x^3 + 1)^{\frac{2}{3}}}.$

20. $y = 3(x^2 + 1)^{\frac{4}{3}}(4x^2 - 3).$

21. $y = 3u^2 - 7.$

22. $y = 4u^3 - 6u^2 + 12u - 3.$

23. $y = (1 - 3u^2 + 6u^4)(1 + u^2)^3.$

24. $y = ux.$

25. $y = u^2 + 3xu^2 + x^4.$

26. $y = \dfrac{u^n}{(a + x)^n}.$

27. $y = u^2x^3w.$

28. Given $(a + x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$; find $(a + x)^4$ by differentiation.

29. Show that the slope of the tangent to the curve $y = x^3$ is never negative. Show where the slope increases or decreases.

30. Given $b^2x^2 + a^2y^2 = a^2b^2$, find $\dfrac{dy}{dx}$: (1) by differentiating as to x; (2) by differentiating as to y; (3) by solving for y and differentiating as to x. Compare the results of the three methods.

31. Show that form (**1**), p. 31, is a special case of (**3**).

32. At what point of the curve $y^2 = ax^3$ is the slope 0? -1? $+1$?

33. Trace the curve $y = x^3 + 3x^2 + x - 1.$

34. $y = \dfrac{3u^2 + 7}{\sqrt{7u^2 + 5}}$ and $u = 5x^2 - 1$; find $\dfrac{dy}{dx}.$

35. At what angle do the curves $y^2 = 12x$ and $y^2 + x^2 + 6x - 63 = 0$ intersect?

20. Elementary transcendental functions.

The following functions are called *transcendental* functions:

Simple exponential functions, consisting of a constant number raised to a power whose exponent is variable, as 4^x, a^{x^2};

general exponential functions, involving a variable raised to a power whose exponent is variable, as $x^{\sin x}$;
the *logarithmic* * functions, as $\log_a x$, $\log_b u$;
the *incommensurable powers* of a variable, as $x^{\sqrt{2}}$, u^{π};
the *trigonometric* functions, as $\sin u$, $\cos u$;
the *inverse trigonometric* functions, as $\sin^{-1} u$, $\tan^{-1} x$.

There are still other transcendental functions, but they will not be considered in this book.

The next four articles treat of the logarithmic, the two exponential functions, and the incommensurable power.

21. Differentiation of $\log_a x$ and $\log_a u$.

Let $$y = \log_a x.$$

Then $$y + \Delta y = \log_a (x + \Delta x),$$

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (x + \Delta x) - \log_a x}{\Delta x}$$

$$= \frac{1}{\Delta x} \log_a \left(\frac{x + \Delta x}{x}\right).$$

For convenience writing h for Δx, and rearranging,

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right)$$

$$= \frac{1}{x} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}};$$

whence $$\frac{dy}{dx} = \frac{1}{x} \lim_{h \doteq 0} \left[\log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}\right].$$

---

* The more general logarithmic function $\log_v u$ is not classified separately, as it can be reduced to the quotient $\frac{\log_a u}{\log_a v}$.

To evaluate the expression $\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$ when $h \doteq 0$, expand it by the binomial theorem, supposing $\frac{x}{h}$ to be a large positive integer m.

The expansion may be written

$$\left(1+\frac{1}{m}\right)^m = 1 + m\cdot\frac{1}{m} + \frac{m(m-1)}{1\cdot 2}\cdot\frac{1}{m^2} + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{m^3} + \cdots,$$

which can be put in the form

$$\left(1+\frac{1}{m}\right)^m = 1 + 1 + \frac{1}{1}\frac{\left(1-\frac{1}{m}\right)}{2} + \frac{1}{1}\frac{\left(1-\frac{1}{m}\right)}{2}\frac{\left(1-\frac{2}{m}\right)}{3} + \cdots.$$

Now as m becomes very large, the terms $\frac{1}{m}, \frac{2}{m}, \cdots$ become very small, and when $m \doteq \infty$ the series approaches the limit

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

The numerical value of this limit can be readily calculated to any desired approximation. This number is an important constant, which is denoted by the letter e, and is equal to 2.7182818 ⋯; thus

$$\lim_{m \doteq \infty}\left(1+\frac{1}{m}\right)^m = e = 2.7182818 \cdots.^*$$

The number e is known as the *natural* or Naperian *base;* and logarithms to this base are called natural or Naperian logarithms. Natural logarithms will be written without a subscript, as $\log x$; in other bases a subscript, as in $\log_a x$, will generally be used to designate the base. The logarithm of e to any base a is called the *modulus* of the system whose base is a.

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* This method of obtaining e is rather too brief to be rigorous; it assumes that $\frac{x}{\Delta x}$ is a positive integer, but that is equivalent to restricting Δx to approach zero in a particular way. It also applies the theorems of limits to the sum and product of an infinite number of terms. The proof is completed on p. 315 of McMahon and Snyder's "Differential Calculus."

If the value, $\lim\limits_{h \doteq 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = e$, be substituted in the expression for $\frac{dy}{dx}$, the result is

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log_a e.$$

More generally, by Art. **14**,

$$\frac{d}{dx} \log_a u = \frac{1}{u} \cdot \log_a e \cdot \frac{du}{dx}. \tag{7}$$

In the particular case in which $a = e$,

$$\frac{d}{dx} \log u = \frac{1}{u} \frac{du}{dx}. \tag{8}$$

The derivative of the logarithm of a function is the product of the derivative of the function and the modulus of the system of logarithms, divided by the function.

22. Differentiation of the simple exponential function.

Let $\qquad y = a^u.$

Then $\qquad \log y = u \log a.$

Differentiating both members of this identity as to x,

$$\frac{1}{y} \frac{dy}{dx} = \log a \cdot \frac{du}{dx}, \text{ by form (8)},$$

$$\frac{dy}{dx} = \log a \cdot y \cdot \frac{du}{dx};$$

therefore

$$\frac{d}{dx} a^u = \log a \cdot a^u \cdot \frac{du}{dx}. \tag{9}$$

In the particular case in which $a = e$,

$$\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}. \tag{10}$$

The derivative of an exponential function with a constant base is equal to the product of the function, the natural logarithm of the base, and the derivative of the exponent.

23. Differentiation of the general exponential function.

Let $$y = u^v,$$

in which u, v are both functions of x.

Take the logarithm of both sides, and differentiate. Then

$$\log y = v \log u,$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}\log u + \frac{v}{u}\frac{du}{dx}, \text{ by forms } (\mathbf{3}), (\mathbf{8}),$$

$$\frac{dy}{dx} = y\left[\log u \cdot \frac{dv}{dx} + \frac{v}{u}\frac{du}{dx}\right];$$

therefore $$\frac{d}{dx}u^v = u^v \log u \frac{dv}{dx} + vu^{v-1}\frac{du}{dx}. \qquad (\mathbf{11})$$

The derivative of an exponential function in which the base is also a variable is obtained by first differentiating, regarding the base as constant, and again, regarding the exponent as constant, and adding the results.

In the differentiation of any given function of this form it is usually better not to substitute in the formula directly but to apply the method just used in deriving (**11**), *i.e.*, to differentiate the logarithm of the function by the preceding rules.

Ex. $y = (4x^2 - 7)^{2+\sqrt{x^2-5}}$, find $\frac{dy}{dx}$.

$$\log y = (2 + \sqrt{x^2 - 5}) \log (4x^2 - 7).$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 5}} \log (4x^2 - 7) + (2 + \sqrt{x^2 - 5})\frac{8x}{4x^2 - 7}.$$

$$\frac{dy}{dx} = (4x^2 - 7)^{2+\sqrt{x^2-5}} \cdot x\left[\frac{\log (4x^2 - 7)}{\sqrt{x^2 - 5}} + \frac{8(2 + \sqrt{x^2 - 5})}{4x^2 - 7}\right].$$

24. Differentiation of an incommensurable power.

Let
$$y = u^n,$$
in which n is an incommensurable constant. Then
$$\log y = n \log u,$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{u} \cdot \frac{du}{dx},$$
$$\frac{dy}{dx} = n \cdot \frac{y}{u} \cdot \frac{du}{dx},$$
$$\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}.$$

This has the same form as (**6**), so that the qualifying word "commensurable" of Art. 19 can now be omitted.

EXERCISES

Find the x derivatives of the following functions:

1. $y = \log (x + a)$.

2. $y = \log (ax + b)$.

3. $y = \log (4x^2 - 7x + 2)$.

4. $y = \log \frac{1+x}{1-x}$.

5. $y = \log \frac{1+x^2}{1-x^2}$.

6. $y = x \log x$.

7. $y = x^n \log x$.

8. $y = x^n \log x^m$.

9. $y = \log \sqrt{1 - x^2}$.

10. $y = \sqrt{x} - \log (\sqrt{x} + 1)$.

11. $y = \log_a (3x^2 - \sqrt{2 + x})$.

12. $y = \log_{10} (x^2 + 7x)$.

13. $y = \log_x a$.

14. $y = e^{xa}$.

15. $y = e^{4x+5}$.

16. $y = e^{\frac{1}{1+x}}$.

17. $y = \frac{e^x}{1 + e^x}$.

18. $y = e^x (1 - x^3)$.

19. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

20. $y = \log (e^x - e^{-x})$.

21. $y = \log (x + e^x)$.

22. $y = x^n a^x$.

23. $y = \log \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$.

24. $y = \frac{1}{\log x}$.

25. $y = (\log x)^2$.

26. $y = \log (\log x)$.

27. $y = x^x$.

28. $y = a^{\log x}$.

The following functions can be easily differentiated by first taking the logarithms of both members of the equations.

29. $y = \dfrac{(x-1)^{\frac{5}{2}}}{(x-2)^{\frac{3}{4}}(x-3)^{\frac{7}{3}}}$.

30. $y = x\sqrt{1-x}\,(1+x)$.

31. $y = \dfrac{x(1+x^2)}{\sqrt{1-x^2}}$.

32. $y = x^5(a+3x)^3(a-2x)^2$.

33. $y = \dfrac{\sqrt{(x+a)^3}}{\sqrt{x-a}}$.

Articles 25–31 will treat of the differentiation of the Trigonometric Functions.

25. Differentiation of sin u.

Let $y = \sin u$.

Then
$$\frac{\Delta y}{\Delta x} = \frac{\sin(u+\Delta u) - \sin u}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$
$$= \frac{2\cos\frac{1}{2}(2u+\Delta u)\sin\frac{1}{2}\Delta u}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$
$$= \cos(u+\tfrac{1}{2}\Delta u)\cdot\frac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\cdot\frac{\Delta u}{\Delta x}.$$

But, when $\Delta u \doteq 0$, $\cos(u+\frac{1}{2}\Delta u) \doteq \cos u$, and $\dfrac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u} \doteq 1$ by Art. 6; hence, passing to the limit,

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}. \tag{12}$$

The derivative of the sine of a function is equal to the product of the cosine of the function and the derivative of the function.

26. Differentiation of cos u.

Let $y = \cos u = \sin\left(\dfrac{\pi}{2} - u\right)$.

Then $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin\left(\dfrac{\pi}{2} - u\right) = \cos\left(\dfrac{\pi}{2} - u\right)\dfrac{d}{dx}\left(\dfrac{\pi}{2} - u\right)$,

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}. \tag{13}$$

The derivative of the cosine of a function is equal to minus the product of the sine of the function and the derivative of the function.

27. Differentiation of tan u.

Let $$y = \tan u = \frac{\sin u}{\cos u}.$$

Then $$\frac{dy}{dx} = \frac{\cos u \cdot \frac{d}{dx}\sin u - \sin u \cdot \frac{d}{dx}\cos u}{\cos^2 u} \qquad \text{by (5)}$$

$$= \frac{\cos^2 u \cdot \frac{du}{dx} + \sin^2 u \cdot \frac{du}{dx}}{\cos^2 u} = \frac{\frac{du}{dx}}{\cos^2 u}, \qquad (12), (13)$$

that is, $$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}. \qquad (14)$$

The derivative of the tangent of a function is equal to the product of the square of the secant of the function and the derivative of the function.

28. Differentiation of cot u.

Let $$y = \cot u = \frac{1}{\tan u}.$$

Then $$\frac{dy}{dx} = \frac{-1}{\tan^2 u} \cdot \frac{d}{dx}\tan u = -\frac{\sec^2 u}{\tan^2 u}\frac{du}{dx}, \qquad (6), (14)$$

$$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}. \qquad (15)$$

The derivative of the cotangent of a function is equal to minus the product of the square of the cosecant of the function and the derivative of the function.

29. Differentiation of sec u.

Let $$y = \sec u = \frac{1}{\cos u}.$$

Then $$\frac{dy}{dx} = \frac{-1}{\cos^2 u} \cdot \frac{d}{dx} \cos u = + \frac{\sin u}{\cos^2 u} \frac{du}{dx},$$

$$\frac{d}{dx} \sec u = \tan u \sec u \frac{du}{dx}. \tag{16}$$

The derivative of the secant of a function is equal to the product of the secant of the function, the tangent of the function, and the derivative of the function.

30. Differentiation of csc u.

Let $$y = \csc u = \frac{1}{\sin u}.$$

Then $$\frac{dy}{dx} = \frac{-1}{\sin^2 u} \cdot \frac{d}{dx} \sin u = - \frac{\cos u}{\sin^2 u} \frac{du}{dx}.$$

$$\frac{d}{dx} \csc u = - \csc u \cot u \frac{du}{dx}. \tag{17}$$

The derivative of the cosecant of a function is equal to minus the product of the cosecant of the function, the cotangent of the function, and the derivative of the function.

31. Differentiation of vers u.

Let $$y = \operatorname{vers} u = 1 - \cos u.$$

Then $$\frac{dy}{dx} = - \frac{d}{dx} \cos u.$$

$$\frac{d}{dx} \operatorname{vers} u = \sin u \frac{du}{dx}. \tag{18}$$

The derivative of the versed-sine of a function is equal to the product of the sine of the function and the derivative of the function.

EXERCISES

Find the x derivatives of the following functions:

1. $y = \sin 7x$.

2. $y = \cos 5x$.

3. $y = \sin x^2$.

4. $y = \sin 2x \cos x$.

5. $y = \sin^3 x$.

6. $y = \sin 5x^2$.

7. $y = \sin^2 7x$.

8. $y = \frac{1}{3}\tan^3 x - \tan x$.

9. $y = \sin^3 x \cos x$.

10. $y = \tan x + \sec x$.

+ **11.** $y = \sin^2 (1 - 2x^2)^2$.

12. $y = \tan (3 - 5x^2)^2$.

13. $y = \tan^2 x - \log (\sec^2 x)$.

+ **14.** $y = \log \tan (\frac{1}{2}x + \frac{1}{4}\pi)$.

15. $y = \log \sin \sqrt{x}$.

+ **16.** $y = \tan a^{\frac{1}{x}}$.

+ **17.** $y = \sin nx \sin^n x$.

18. $y = \sin (u + b) \cos (u - b)$. ✓

19. $y = \frac{\sin^m nx}{\cos^n mx}$. ✓

20. $y = x + \log \cos\left(x - \frac{\pi}{4}\right)$.

21. $y = \log \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$.

22. $y = \sin (\sin u)$.

23. $y = \sin^2 e^{ax}$.

24. $y = \sin e^x \cdot \log x$.

25. $y = \sqrt{\sin x^2}$. ✓

26. $y = x^{\sin x}$.

27. $y = \csc^2 4x$. ✓

28. $y = \sec (4x - 3)^2$. ✓

29. $y = \text{vers}\, x^3$. ✓

30. $y = \cot x^2 + \sec \sqrt{x}$. ✓

31. $y = \sin xy$.

32. $y = \tan (x + y)$.

32. Differentiation of $\sin^{-1} u$.

Let $y = \sin^{-1} u$.

Then $\sin y = u$,

and, by differentiating both members of this identity,

$$\cos y \frac{dy}{dx} = \frac{du}{dx};$$

hence,

$$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} = \frac{1}{\pm\sqrt{1 - \sin^2 y}}\frac{du}{dx};$$

i.e.,

$$\frac{d}{dx}\sin^{-1} u = \pm\frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}.$$

The ambiguity of sign accords with the fact that $\sin^{-1} u$ is a many-valued function of u, since, for any value of u between -1 and 1, there is a series of angles whose sine is u: and, when u receives an increase, some of these angles increase and some decrease; hence, for some of them, $\frac{d\sin^{-1}u}{du}$ is positive, and for some negative. It will be seen that, when $\sin^{-1}u$ lies in the first or fourth quarter, it increases with u, and, when in the second or third, it decreases as u increases. Hence, for the angles of the first and fourth quarters,

$$\frac{d}{du}\sin^{-1}u = +\frac{1}{\sqrt{1-u^2}},\quad \frac{d}{dx}\sin^{-1}u = +\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}. \tag{19}$$

In the other quarters the minus sign is to be used before the radical.

33. Differentiation of the remaining inverse trigonometric forms.

The derivatives of the other inverse trigonometric functions can be easily obtained by the method employed in the last article. The results are as follows:

$$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \quad \text{(in 1st and 2d quarters).} \tag{20}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx} \quad \text{(in all quarters).} \tag{21}$$

$$\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\frac{du}{dx} \quad \text{(in all quarters).} \tag{22}$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \quad \text{(in 1st and 3d quarters).} \tag{23}$$

$$\frac{d}{dx}\csc^{-1}u = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx} \quad \text{(in 1st and 3d quarters).} \tag{24}$$

$$\frac{d}{dx}\operatorname{vers}^{-1}u = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx} \quad \text{(in 1st and 2d quarters).} \tag{25}$$

The radicals in forms (20), (23), (24), (25) receive the opposite signs respectively when the angles are taken in the quarters other than those stated.

EXERCISES

Find the x-derivative of each of the following functions:

1. $y = \sin^{-1} 2x^2$.

2. $y = \cos^{-1}\sqrt{1 - x^2}$.

3. $y = \sin^{-1}(3x - 1)$.

4. $y = \sin^{-1}(3x - 4x^3)$.

5. $y = \sin^{-1}\dfrac{1 - x^2}{1 + x^2}$.

6. $y = \sqrt{\sin^{-1} x}$.

7. $y = \tan^{-1} e^x$.

8. $y = \cos^{-1} \log x$.

9. $y = \sin^{-1}(\tan x)$.

10. $y = \sec^{-1}\dfrac{1}{\sqrt{1 - x^2}}$.

11. $y = \operatorname{vers}^{-1} 2x^2$.

12. $y = \tan^{-1}\left(\dfrac{1}{\sqrt{x^2 - 1}}\right)$.

13. $y = \operatorname{vers}^{-1}\dfrac{2x^2}{1 + x^2}$.

14. $y = \sin^{-1}\sqrt{\sin x}$.

15. $y = \tan^{-1}\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$.

16. $y = \tan x \cdot \tan^{-1} x$.

17. $y = x \sin^{-1} x$.

18. $y = e^{\tan^{-1} x}$.

19. $y = \csc^{-1}\dfrac{1}{2x^2 - 1}$.

20. $y = \sec^{-1}\dfrac{x^2 + 1}{x^2 - 1}$.

21. $y = \tan^{-1}\dfrac{\sqrt{x} + \sqrt{a}}{1 - \sqrt{ax}}$.

22. $y = \cos^{-1}\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

23. $y = \tan^{-1}(n \tan x)$.

24. $y = \cos^{-1}(\cos 2x)$.

25. $y = \cos^{-1}(2 \cos x)$.

26. $y = \tan^{-1}(\sqrt{1 + x^2} - x)$.

27. $y = 2 \tan^{-1}\sqrt{\dfrac{1 - x}{1 + x}}$.

28. $y = \tan^{-1}\dfrac{2x - b}{b\sqrt{3}} + \tan^{-1}\dfrac{2b - x}{x\sqrt{3}}$.

34. Table of fundamental forms.

$$\frac{d(cu)}{dx} = c\frac{du}{dx}. \tag{1}$$

$$\frac{d}{dx}(u + v + w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}. \tag{2}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3}$$

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}. \tag{4}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{5}$$

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{6}$$

$$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}. \tag{7}$$

$$\frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}. \tag{8}$$

$$\frac{d}{dx}a^u = \log_e a \cdot a^u \cdot \frac{du}{dx}. \tag{9}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}. \tag{10}$$

$$\frac{d}{dx}u^v = u^v \cdot \log_e u \cdot \frac{dv}{dx} + vu^{v-1}\frac{du}{dx}. \tag{11}$$

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}. \tag{12}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}. \tag{13}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}. \tag{14}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}. \tag{15}$$

$$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}. \tag{16}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u\frac{du}{dx}. \tag{17}$$

$$\frac{d}{dx}\operatorname{vers} u = \sin u\frac{du}{dx}. \tag{18}$$

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{19}$$

$$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}. \tag{20}$$

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}. \tag{21}$$

$$\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx}. \tag{22}$$

$$\frac{d}{dx}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}. \tag{23}$$

$$\frac{d}{dx}\csc^{-1} u = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}. \tag{24}$$

$$\frac{d}{dx}\operatorname{vers}^{-1} u = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx}. \tag{25}$$

EXERCISES ON CHAPTER II

Find the x derivatives of the following functions:

1. $y = 3x^2 + 5x^3 - 7.$

2. $y = \frac{3}{x^2} + \frac{5}{x^3} - \frac{1}{7}.$

3. $y = (x+5)\sqrt{x-3}.$

4. $y = x\sqrt{a^2 - x^2}.$

5. $y = x \log \sin x.$

6. $y = \frac{a}{x}\sqrt{a^2 - x^2}.$

7. $y = \frac{c^2}{x} e^{\frac{x}{c}}.$

8. $y = \tan 2z,\ z = \tan^{-1}(2x - 1).$

9. $y = e^{\sqrt{u}},\ u = \log \sin x.$

10. $y = \log \frac{x}{a^x}.$

11. $y = \frac{1 - x^2}{\sqrt{1 + x^2}}.$

12. $y = e^x \cos x.$

13. $y = \operatorname{vers}^{-1}\left(\frac{1}{x}\right).$

14. $y = \tan^{-1}\frac{4 \sin x}{3 + 5 \cos x}.$

15. $y = (x + a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax}.$

16. $y = \cot^{-1}\frac{1 + \sqrt{1 + x^2}}{x}.$

17. $y = \tan^4 x - 2\tan^2 x + \log(\sec^4 x).$

18. $y = \frac{x \log x}{1 - x} + \log(1 - x).$

19. $y = \cos^{-1}\frac{3 + 5\cos x}{5 + 3 \cos x}.$

20. $y = \log\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1} x.$

21. $y = \log(x + \sqrt{x^2 - a^2}) + \sec^{-1}\frac{x}{a}.$

22. $y = e^u,\ u = \log x.$

23. $y = \log s^2 + e^s,\ s = \sec x.$

24. $x^3 + y^3 - 3axy = 0.$

25. $x^2y^2 + x^3 + y^3 = 0.$

26. $x^3 + x = y + y^3.$

27. $xy^2 + x^2y = x + y.$

28. $y = \sin(2u - 7),\ u = \log x^2.$

29. For what values of x is the function $\frac{x^3}{a^2 + x^2}$ an increasing function?

30. Prove that $\tan^{-1}\left(\frac{\sqrt{1 + x^2} - 1}{x}\right)$ always increases with x.

31. Show that the x-derivative of $\tan^{-1}\sqrt{\frac{1 - \cos x}{1 + \cos x}}$ is not a function of x.

32. Find at what points of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the tangent cuts off equal intercepts on the axes.

33. Find the points at which the slope of the curve $y = \tan x$ is twice that of the line $y = x$.

34. Find the angle which the curves $y = \sin x$ and $y = \cos x$ make with each other at their point of intersection.

CHAPTER III

SUCCESSIVE DIFFERENTIATION

35. Definition of *n*th derivative. When a given function $y = \phi(x)$ is differentiated with regard to x by the rules of Chapter I, then the result

$$\frac{dy}{dx} = \phi'(x)$$

is a new function of x which may itself be differentiated by the same rules. Thus,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\phi'(x).$$

The left-hand member is usually abbreviated to $\frac{d^2y}{dx^2}$, and the right-hand member to $\phi''(x)$; that is,

$$\frac{d^2y}{dx^2} = \phi''(x).$$

Differentiating again and using a similar notation,

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = \phi'''(x),$$

and so on for any number of differentiations. Thus the symbol $\frac{d^2y}{dx^2}$ expresses that y is to be differentiated with regard to x, and that the resulting derivative is then to be differentiated. Similarly, $\frac{d^3y}{dx^3}$ indicates the performance of the operation $\frac{d}{dx}$ three times, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$. In general, the symbol $\frac{d^ny}{dx^n}$ means that y is to be differentiated n times in succession with regard to x.

Ex. **1.** If

$$y = x^4 + \sin 2\,x,$$

$$\frac{dy}{dx} = 4\,x^3 + 2\cos 2\,x,$$

$$\frac{d^2y}{dx^2} = 12\,x^2 - 4\sin 2\,x,$$

$$\frac{d^3y}{dx^3} = 24\,x - 8\cos 2\,x,$$

$$\frac{d^4y}{dx^4} = 24 + 16\sin 2\,x.$$

If an implicit equation between x and y be given and the derivatives of y with regard to x are required, it is not necessary to solve the equation for either variable before performing the differentiation.

Ex. **2.** Given $x^4+y^4+4\,a^2xy=0$; find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}(x^4 + y^4 + 4\,a^2xy) = 0,$$

$$\frac{d}{dx}x^4 + \frac{d}{dx}y^4 + 4\,a^2\frac{d}{dx}xy = 0,$$

$$4\,x^3 + 4\,y^3\frac{dy}{dx} + 4\,a^2x\frac{dy}{dx} + 4\,a^2y = 0.$$

The last equation is now to be solved for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{x^3 + a^2y}{y^3 + a^2x}. \tag{1}$$

Differentiating again,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x^3 + a^2y}{y^3 + a^2x}\right)$$

$$= -\frac{(y^3 + a^2x)\frac{d}{dx}(x^3 + a^2y) - (x^3 + a^2y)\frac{d}{dx}(y^3 + a^2x)}{(y^3 + a^2x)^2}$$

$$= -\frac{(y^3 + a^2x)\left(3\,x^2 + a^2\frac{dy}{dx}\right) - (x^3 + a^2y)\left(3\,y^2\frac{dy}{dx} + a^2\right)}{(y^3 + a^2x)^2}.$$

The value of $\frac{dy}{dx}$ from (1) is now to be substituted in the last equation, and the resulting expression simplified. The final form may be written:

$$\frac{d^2y}{dx^2} = \frac{2a^6xy - 10a^2x^3y^3 - a^4(x^4 + y^4) - 3x^2y^2(x^4 + y^4)}{(y^3 + a^2x)^3}.$$

In like manner higher derivatives may be found.

36. Expression for the *n*th derivative in certain cases. For certain functions, a general expression for the *n*th derivative can be readily obtained in terms of *n*.

Ex. 1. If $y = e^x$, then $\frac{dy}{dx} = e^x$, $\frac{d^2y}{dx^2} = e^x$, $\cdots$, $\frac{d^ny}{dx^n} = e^x$,

where n is any positive integer. If $y = e^{ax}$, $\frac{d^ny}{dx^n} = a^ne^{ax}$.

Ex. 2. If

$$y = \sin x,$$

$$\frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right).$$

If $y = \sin ax$, $\frac{d^ny}{dx^n} = a^n \sin\left(ax + n\frac{\pi}{2}\right)$.

EXERCISES ON CHAPTER III

1. $y = 3x^4 + 5x^2 + 3x - 9$; find $\frac{d^3y}{dx^3}$.

2. $y = 2x^2 + 3x + 5$; find $\frac{d^3y}{dx^3}$.

3. $y = \frac{1}{x}$; find $\frac{d^3y}{dx^3}$.

4. $y = x^2 - \frac{1}{x^2}$; find $\frac{d^4y}{dx^4}$.

5. $y = e^x$; find $\frac{d^5y}{dx^5}$.

6. $y = e^x \log x$; find $\frac{d^2y}{dx^2}$.

7. $y = x^2 \log x$; find $\frac{d^2y}{dx^2}$.

✓ **8.** $y = \sec^2 x$; find $\frac{d^3y}{dx^3}$.

9. $y = \log \sin x$; find $\frac{d^3y}{dx^3}$.

10. $y = \sin x \cos x$; find $\frac{d^4y}{dx^4}$.

11. $y = \dfrac{x^3}{1-x}$; find $\dfrac{d^4y}{dx^4}$.

12. $y = x^4 \log x^2$; find $\dfrac{d^5y}{dx^5}$.

13. $y = \sin x$; find $\dfrac{d^{12}y}{dx^{12}}$.

14. $y = \log (e^x + e^{-x})$; find $\dfrac{d^3y}{dx^3}$.

15. $y = (x^2 - 3x + 3)\, e^{2x}$; find $\dfrac{d^3y}{dx^3}$.

16. $y = x^4 \log x$; find $\dfrac{d^6y}{dx^6}$.

17. $y = e^{ax}$; find $\dfrac{d^ny}{dx^n}$.

18. $y = \dfrac{1}{x-1}$; find $\dfrac{d^ny}{dx^n}$.

19. $y = \cos mx$; find $\dfrac{d^ny}{dx^n}$.

Solve

20. $y = \dfrac{1}{(a+x)^m}$; find $\dfrac{d^ny}{dx^n}$.

21. $y = \log (a+x)^m$; find $\dfrac{d^ny}{dx^n}$.

22. $y^2 = 2px$; find $\dfrac{d^3y}{dx^3}$.

23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; find $\dfrac{d^2y}{dx^2}$.

24. $x^3 + y^3 = 3axy$; find $\dfrac{d^2y}{dx^2}$.

25. $e^{x+y} = xy$; find $\dfrac{d^2y}{dx^2}$.

26. $y = 1 + xe^y$; find $\dfrac{d^2y}{dx^2}$.

27. $y = e^x \sin x$; prove $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

28. $y = ax \sin x$; prove $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + (x^2+2)\,y = 0$.

29. $y = ax^{n+1} + bx^{-n}$; prove $x^2 \dfrac{d^2y}{dx^2} = n(n+1)\,y$.

30. $y = (\sin^{-1} x)^2$; prove $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$.

31. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$; prove $\dfrac{dy}{dx} = 1 - y^2$.

32. $y = \dfrac{1}{4x^2 - 1}$; find $\dfrac{d^ny}{dx^n}$.

33. $y = x^{n-1} \log x$; find $\dfrac{d^ny}{dx^n}$.

34. $y = \dfrac{1-x}{1+x}$; find $\dfrac{d^ny}{dx^n}$.

35. $y = x^2 e^x$; prove $\dfrac{d^ny}{dx^n} = 2\dfrac{d^{n-1}y}{dx^{n-1}} - \dfrac{d^{n-2}y}{dx^{n-2}} + 2e^x$.

36. $y = \cos^2 x$; find $\dfrac{d^ny}{dx^n}$.

37. From the relation $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{-\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}$.

CHAPTER IV

EXPANSION OF FUNCTIONS

It is sometimes necessary to expand a given function in a series of powers of the independent variable. For instance, in order to compute and tabulate the successive numerical values of sin x for different values of x, it is convenient to have sin x developed in a series of powers of x with coefficients independent of x.

Simple cases of such development have been met with in algebra. For example, by the binomial theorem,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \cdots; \tag{1}$$

and again, by ordinary division,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots. \tag{2}$$

It is to be observed, however, that the series is a proper representative of the function only for values of x within a certain interval. For instance, the identity in (1) holds only for values of x between $-a$ and $+a$ when n is not a positive integer; and the identity in (2) holds only for values of x between -1 and $+1$. In each of these examples, if a finite value outside of the stated limits be given to x, the sum of an infinite number of terms of the series will be infinite, while the function in the first member will be finite.

37. Convergence and divergence of series.* An infinite series is said to be *convergent* or *divergent* according as the sum of the first n terms of the series does or does not approach a finite limit when n is increased without limit.

Those values of x for which a series of powers of x is convergent constitute the *interval of convergence* of the series.

For example, the sum of the first n terms of the geometric series

$$a + ax + ax^2 + ax^3 + \cdots$$

is

$$s_n = \frac{a(1 - x^n)}{1 - x}.$$

First let x be numerically less than unity. Then when n is taken sufficiently large, the term x^n approaches zero;

hence

$$\lim_{n \doteq \infty} s_n = \frac{a}{1 - x}.$$

Next let x be numerically greater than unity. Then x^n becomes infinite when n is infinite; hence, in this case

$$\lim_{n \doteq \infty} s_n = \infty.$$

Thus the given series is convergent or divergent according as x is numerically less or greater than unity. The condition for convergence may then be written

$$-1 < x < 1,$$

and the interval of convergence is between -1 and $+1$.

Similarly the geometric series

$$1 - 3x + 9x^2 - 27x^3 + \cdots,$$

---

* For an elementary, yet comprehensive and rigorous, treatment of this subject, see Professor Osgood's "Introduction to Infinite Series" (Harvard University Press, 1897).

whose common ratio is $-3x$, is convergent or divergent according as $3x$ is numerically less or greater than unity.

The condition for convergence is $-1 < 3x < 1$, and hence the interval of convergence is between $-\frac{1}{3}$ and $+\frac{1}{3}$.

38. General test for interval of convergence. The following summary of algebraic principles leads up to a test that is sufficient to find the interval of convergence for a series of the most usual kind, that is, a series consisting of positive integral powers of x, in which the coefficient of x^n is a known function of n.

1. If s_n is a variable that continually increases with n, but for all values of n remains less than some fixed number k, then s_n approaches some definite limit not greater than k. [This follows from the definition of a limit.]

2. If one series of positive terms is known to be convergent, and if the terms of another series be positive and less than the corresponding terms of the first series, then the latter series is convergent. [Use 1.]

3. If, after a given term, the terms of a series form a decreasing geometric progression, then:

(*a*) The successive terms approach nearer and nearer to zero as a limit;

(*b*) The sum of all the terms approaches some fixed constant as a limit. [Use method of last article.]

4. If the terms of a series be positive, and if after a given term the ratio of each term to the preceding be less than a fixed proper fraction, the series is convergent. [Use 2 and 3.]

5. If there be a series A consisting of an infinite number of both positive and negative terms, and if another series B, obtained therefrom by making all the terms positive, is known to be convergent, then the series A is convergent.

For the positive terms of A must form a convergent series, otherwise the series B could not be convergent; similarly the negative terms of A must form a convergent series. Let the sums of these convergent series be u and $-v$. Let the first n terms of series A contain m positive terms and p negative terms. Let Σ_m, $-T_p$, S_n denote the sum of the positive terms, the sum of the negative terms, and the sum of all n terms respectively. Then $S_n = \Sigma_m - T_p$. Now when n approaches infinity, m and p also approach infinity and hence

$$\lim_{n \doteq \infty} S_n = \lim_{m \doteq \infty} \Sigma_m - \lim_{p \doteq \infty} T_p, \quad i.e.,\ S = u - v.$$

Therefore the series A is convergent.

DEFINITIONS. The *absolute value* of a real number x is its numerical value taken positively, and is written $|x|$. The equation $|x| = |a|$ indicates that the absolute value of x is equal to the absolute value of a. When, however, x and a are replaced by longer expressions, it is convenient to write the relation in the form $x| = |a$, in which the symbol $| = |$ is read "equals in absolute value." In like manner, the symbols $| < |$, $| > |$ will be used to indicate that the expression on the left has respectively a smaller, or larger, numerical value than the one on the right.

Any series of terms is said to be *absolutely* or *unconditionally* convergent when the series formed by their absolute values is convergent. When a series is convergent, but the series formed by making each term positive is not convergent, the first series is said to be *conditionally* convergent.*

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* The appropriateness of this terminology is due to the fact that the terms of an absolutely convergent series can be rearranged in any way, without altering the limit of the sum of the series; and that this is not true of a conditionally

E.g., the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$ is absolutely convergent; but the series $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots$ is conditionally convergent.

6. If there be any series of terms in which after some fixed term the ratio of each term to the preceding is numerically less than a fixed proper fraction; then,

(*a*) the successive terms of the series approach nearer and nearer to zero as a limit;

(*b*) the sum of all the terms approaches some fixed constant as a limit; and the series is absolutely convergent. [Use 3, 4, 5.]

Ex. **1**. Find the interval of convergence of the series

$$1+2\cdot 2x+3\cdot 4x^2+4\cdot 8x^3+5\cdot 16x^4+\cdots.$$

Here the nth term u_n is $n2^{n-1}x^{n-1}$, and the $(n+1)$st term u_{n+1} is $(n+1)2^nx^n$, hence

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)2^nx^n}{n2^{n-1}x^{n-1}}=\frac{(n+1)2x}{n},$$

therefore when $n \doteq \infty$, $$\frac{u_{n+1}}{u_n}\doteq 2x.$$

It follows by (6) that the series is absolutely convergent when $-1<2x<1$, and that the interval of convergence is between $-\frac{1}{2}$ and $+\frac{1}{2}$. The series is evidently not convergent when x has either of the extreme values.

Ex. **2**. Find the interval of convergence of the series

$$\frac{x}{1\cdot 3}-\frac{x^3}{3\cdot 3^3}+\frac{x^5}{5\cdot 3^5}-\frac{x^7}{7\cdot 3^7}+\cdots+\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)3^{2n-1}}+\cdots.$$

Here $$\frac{u_{n+1}}{u_n}\Big|=\Big|\frac{2n-1}{2n+1}\cdot\frac{3^{2n-1}}{3^{2n+1}}\cdot\frac{x^{2n+1}}{x^{2n-1}}=\frac{2n-1}{2n+1}\cdot\frac{x^2}{3^2},$$

hence $$\frac{u_{n+1}}{u_n}\doteq\frac{x^2}{3^2},\ \text{when } n\doteq\infty;$$

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convergent series. Thus the numerical value of the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$ is independent of the order or grouping; but the value of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ can be made equal to any number whatever by suitable rearrangement. [For a simple proof, see Osgood, pp. 43, 44.]

thus the series is absolutely convergent when $\frac{x^2}{3^2} < 1$, *i.e.*, when $-3 < x < 3$, and the interval of convergence is from -3 to $+3$. The extreme values of x, in the present case, render the series conditionally convergent.

Ex. **3**. Show that the series $\frac{1}{1^2}\left(\frac{x}{3}\right) - \frac{1}{3^2}\left(\frac{x}{3}\right)^3 + \frac{1}{5^2}\left(\frac{x}{3}\right)^5 - \frac{1}{7^2}\left(\frac{x}{3}\right)^7 + \cdots$ has the same interval of convergence as the last; but that the extreme values of x render the series absolutely convergent.

39. Remainder after *n* terms. The last article treated of the interval of convergence of a given series without reference to the question whether or not it was the development of any known function. On the other hand, the series that present themselves in this chapter are the developments of given functions, and the first question that arises is concerning those values of x for which the function is equivalent to its development.

When a series has such a generating function, the difference between the value of the function and the sum of the first n terms of its development is called the ***remainder after n terms***. Thus if $f(x)$ be the function, $S_n(x)$ the sum of the first n terms of the series, and $R_n(x)$ the remainder obtained by subtracting $S_n(x)$ from $f(x)$, then

$$f(x) = S_n(x) + R_n(x),$$

in which $S_n(x)$, $R_n(x)$ are functions of n as well as of x.

If $$\lim_{n \doteq \infty} R_n(x) = 0, \text{ then } \lim_{n \doteq \infty} S_n(x) = f(x);$$

thus the limit of the series $S_n(x)$ is the generating function when the limit of the remainder is zero. Frequently this is a sufficient test for the convergence of a series $\lim\limits_{n \doteq \infty} S_n(x)$.

If a series proceed in integral powers of $x - a$, the preceding conditions are to be modified by substituting $x - a$ for x; otherwise each criterion is to be applied as before.

40. Maclaurin's expansion of a function in a power-series.* It will now be shown that all the developments of functions in power-series given in algebra and trigonometry are but special cases of one general formula of expansion.

It is proposed to find a formula for the expansion, in ascending positive integral powers of $x - a$, of any assigned function which, with its successive derivatives, is continuous in the vicinity of the value $x = a$.

The preliminary investigation will proceed on the hypothesis that the assigned function $f(x)$ has such a development, and that the latter can be treated as identical with the former for all values of x within a certain interval of equivalence that includes the value $x = a$. From this hypothesis the coefficients of the different powers of $x - a$ will be determined. It will then remain to test the validity of the result by finding the conditions that must be fulfilled, in order that the series so obtained may be a proper representation of the generating function.

Let the assumed identity be

$$f(x) \equiv A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + \cdots, \quad (1)$$

in which A, B, C, $\cdots$ are undetermined coefficients independent of x.

Successive differentiation with regard to x supplies the following additional identities, on the hypothesis that the derivative of each series can be obtained by differentiating it term by term, and that it has some interval of equivalence with its corresponding function:

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* Named after Colin Maclaurin (1698-1746), who published it in his "Treatise on Fluxions" (1742); but he distinctly says it was known by Stirling (1690-1772), who also published it in his "Methodus Differentialis" (1730), and by Taylor (see Art. 41).

$$\begin{aligned}
f'(x) &= B + 2C(x-a) + 3D(x-a)^2 + 4E(x-a)^3 + \cdots\\
f''(x) &= 2C + 3\cdot 2D(x-a) + 4\cdot 3E(x-a)^2 + \cdots\\
f'''(x) &= 3\cdot 2D + 4\cdot 3\cdot 2E(x-a) + \cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

If, now, the special value a be given to x, the following equations will be obtained:

$$f(a) = A,\ f'(a) = B,\ f''(a) = 2C,\ f'''(a) = 3\cdot 2D,\ \cdots.$$

Hence,

$$A = f(a),\ B = f'(a),\ C = \frac{f''(a)}{2!},\ D = \frac{f'''(a)}{3!},\ \cdots.$$

Thus the coefficients in (1) are determined, and the required development is

$$\boldsymbol{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$$
$$\boldsymbol{+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.} \qquad (2)$$

This series is known as *Maclaurin's* series, and the theorem expressed in the formula is called Maclaurin's theorem.

Ex. **1.** Expand $\log x$ in powers of $x - a$.

Here $f(x) = \log x,\ f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{1\cdot 2}{x^3}\cdots,$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Hence, $f(a) = \log a,\ f'(a) = \frac{1}{a},\ f''(a) = -\frac{1}{a^2},\ f'''(a) = \frac{1\cdot 2}{a^3}\cdots,$

$$f^{(n)}(a) = \frac{(-1)^{n-1}(n-1)!}{a^n},$$

and, by (2), the required development is

$$\log x = \log a + \frac{1}{a}(x-a) - \frac{1}{2a^2}(x-a)^2 + \frac{1}{3a^3}(x-a)^3 - \cdots$$
$$+ \frac{(-1)^{n-1}}{na^n}(x-a)^n + \cdots.$$

The condition for the convergence of this series is

$$\lim_{n \doteq \infty}\left[\frac{(x-a)^{n+1}}{(n+1)a^{n+1}} : \frac{(x-a)^n}{na^n}\right] |<| 1;$$

i.e.,

$$\frac{x-a}{a} |<| 1,$$

$$x - a |<| a,$$

$$0 < x < 2a.$$

This series may be called the *development of* $\log x$ *in the vicinity of* $x = a$. Its development in the vicinity of $x = 1$ has the simpler form

$$\log x = x - 1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots,$$

which holds for values of x between 0 and 2.

Ex. **2**. Show that the development of $\frac{1}{x}$ in powers of $x - a$ is

$$\frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \cdots,$$

and that the series is convergent from $x = 0$ to $x = 2a$.

Ex. **3**. Develop e^x in powers of $x - 2$.

Ex. **4**. Develop $x^3 - 2x^2 + 5x - 7$ in powers of $x - 1$.

Ex. **5**. Develop $3y^2 - 14y + 7$ in powers of $y - 3$.

The expansion of a function $f(x)$ in a series of ascending powers of x can be obtained at once from formula (2) by giving a the particular value zero. The series then becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \qquad (3)$$

Ex. **6**. Expand $\sin x$ in powers of x, and find the interval of convergence of the series.

Here $f(x) = \sin x,$ $\qquad f(0) = 0,$

$f'(x) = \cos x,$ $\qquad f'(0) = 1,$

$f''(x) = -\sin x,$ $\qquad f''(0) = 0,$

$f'''(x) = -\cos x,$ $\qquad f'''(0) = -1,$

$f^{\text{iv}}(x) = \sin x,$ $\qquad f^{\text{iv}}(0) = 0,$

$f^{\text{v}}(x) = \cos x,$ $\qquad f^{\text{v}}(0) = 1,$

Hence, by (3),

$$\sin x = 0 + 1 \cdot x + \frac{0 \cdot x^2}{2!} - \frac{1}{3!}x^3 + \frac{0 \cdot x^4}{4!} + \frac{1}{5!}x^5 \cdots;$$

thus the required development is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + \cdots.$$

To find the interval of convergence of the series, use the method of Art. 38. The ratio of u_{n+1} to u_n becomes

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{(2n+1)!} : \frac{x^{2n-1}}{(2n-1)!}\right| = \frac{x^2}{(2n+1)\,2n}.$$

This ratio approaches the limit zero, when n becomes infinite, however large be the constant value assigned to x. This limit being less than unity, the series is convergent for any finite value of x, and hence the interval of convergence is from $-\infty$ to $+\infty$.

The preceding series may be used to compute the numerical value of $\sin x$ for any given value of x. Take, for example, $x = .5$ radians. Then

$$\sin(.5) = .5 - \frac{(.5)^3}{2 \cdot 3} + \frac{(.5)^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{(.5)^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots,$$

$$\begin{aligned} &= .5000000 \\ &- .0208333 \\ &+ .0002604 \\ &- .0000015 \\ &+ .0000000 \\ \hline \sin(.5) &= .4794256 \cdots \end{aligned}$$

Show that the ratio of u_5 to u_4 is $\frac{1}{288}$; and hence that the error in stopping at u_4 is numerically less than $u_4[\frac{1}{288} + (\frac{1}{288})^2 + \cdots]$, that is, $< \frac{1}{287}u_4$.

Ex. 7. Show that the development of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} + \cdots,$$

and that the interval of convergence is from $-\infty$ to $+\infty$.

Ex. 8. Develop the exponential functions a^x, e^x.

Here

$$f(x) = a^x,\ f'(x) = a^x \log a,\ f''(x) = a^x(\log a)^2,\ \cdots,\ f^{(n)}(x) = a^x(\log a)^n;$$

hence $f(0) = 1,\ f'(0) = \log a,\ f''(0) = (\log a)^2,\ \cdots,\ f^{(n)}(0) = (\log a)^n,$

and $$a^x = 1 + (\log a)\,x + \frac{(\log a)^2}{2!}x^2 + \cdots + \frac{(\log a)^n}{n!}x^n + \cdots.$$

As a special case, put $a = e$.

Then $$\log a = \log e = 1,$$

and $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

These series are convergent for every finite value of x.

41. Taylor's series. If a function of the sum of two numbers a and x be given, $f(a + x)$, it is frequently desirable to expand the function in powers of one of them, say x.

In the function $f(a + x)$, a is to be regarded as constant, so that, considered as a function of x, it may be expanded by formula (3) of the preceding article. In that formula, the constant term in the expansion is the value which the function has when x is made equal to zero, hence the first term in the expansion of $f(a + x)$ may be written $f(a)$. In the same manner the coefficients of the successive powers of x are the corresponding derivatives of $f(a + x)$ as to x, in which x is put equal to zero after the differentiation has been performed. The expansion may therefore be written

$$\boldsymbol{f(a + x) = f(a) + f'(a)\,x + \frac{f''(a)}{2!}x^2 + \cdots + \frac{f^{(n)}(a)}{n!}x^n + \cdots.}$$

This series, from the name of its discoverer, is known as *Taylor's series,* and the theorem expressed by the formula is known as Taylor's theorem.

Ex. Expand $\sin(a + x)$ in powers of x.

Here $$f(a + x) = \sin(a + x),$$

hence $$f(a) = \sin a,$$

and $$f'(a) = \cos a,$$

. . . .

Hence $\sin(a + x) = \sin a + \cos a \cdot x - \dfrac{\sin a}{2!}x^2 - \dfrac{\cos a}{3!}x^3 + \cdots.$

EXERCISES

1. Expand $\tan x$ in powers of x.

✓ 2. Compare the expansion of $\tan x$ with the quotient derived by dividing the series for $\sin x$ by that for $\cos x$.

See Exs. 6 and 7, Art. 40.

3. Prove $\log \frac{\sin x}{x} = -\frac{x^2}{6} - \frac{x^4}{180} \cdots$.

expand sin x first, then log $[1-(\frac{x^2}{3!}-\frac{x^4}{5!})]$

4. Prove $\log(x+\sqrt{1+x^2}) = x - \frac{x^3}{2 \cdot 3} + \frac{3}{2 \cdot 4}\frac{x^5}{5} - \cdots$.

✓ 5. Prove $\log \cos x = -\frac{x^2}{2} - \frac{2x^4}{4!} - \frac{16x^6}{6!} - \frac{272x^8}{8!} \cdots$.

6. Expand $\frac{e^x}{1+x}$ by division, making use of the exponential series.

7. Find the expansion of $e^x \log(1+x)$ to the term involving x^5, by multiplying together a sufficient number of terms of the series for e^x and for $\log(1+x)$.

✓ 8. Expand $\sqrt{1-x^2}$ in powers of x.

✓ 9. Expand $\frac{1+x}{1-x}$ in powers of x.

10. Arrange $(3+x)^4 - 5(3+x)^3 + 2(3+x)^2 - (3+x) - 2$ in powers of x.

42. A necessary restriction imposed upon the series so that it may be a correct representative of the generating function, is that the remainder after n terms may be made smaller than any given number by taking n large enough.

Before deriving the general form for this remainder it is necessary to prove the following theorem.

Rolle's theorem. If $f(x)$ and its first derivative are continuous for all values of x between a and b, and if $f(a), f(b)$ both vanish, then $f'(x)$ will vanish for some value of x between a and b.

By supposition $f(x)$ cannot become infinite for any value of x, such that $a < x < b$. If $f'(x)$ does not vanish, it must always be positive or always be negative; hence, $f(x)$ must continually increase or continually decrease as x increases from a to b (Art. 13).

This is impossible, since by hypotheses $f(a) = 0$ and $f(b) = 0$; hence, at some point x between a and b, $f(x)$ must cease to increase and begin to decrease, or cease to decrease and begin to increase.

This point x is defined by the equation $f'(x) = 0$.

To prove the same thing geometrically, let $y = f(x)$ be the equation of a continuous curve, which crosses the x-axis at distances $x = a$, $x = b$ from the origin. Then at some point P between a and b the tangent to the curve is parallel to the x-axis, since by supposition there is no discontinuity in the slope of the tangent. Hence at the point P

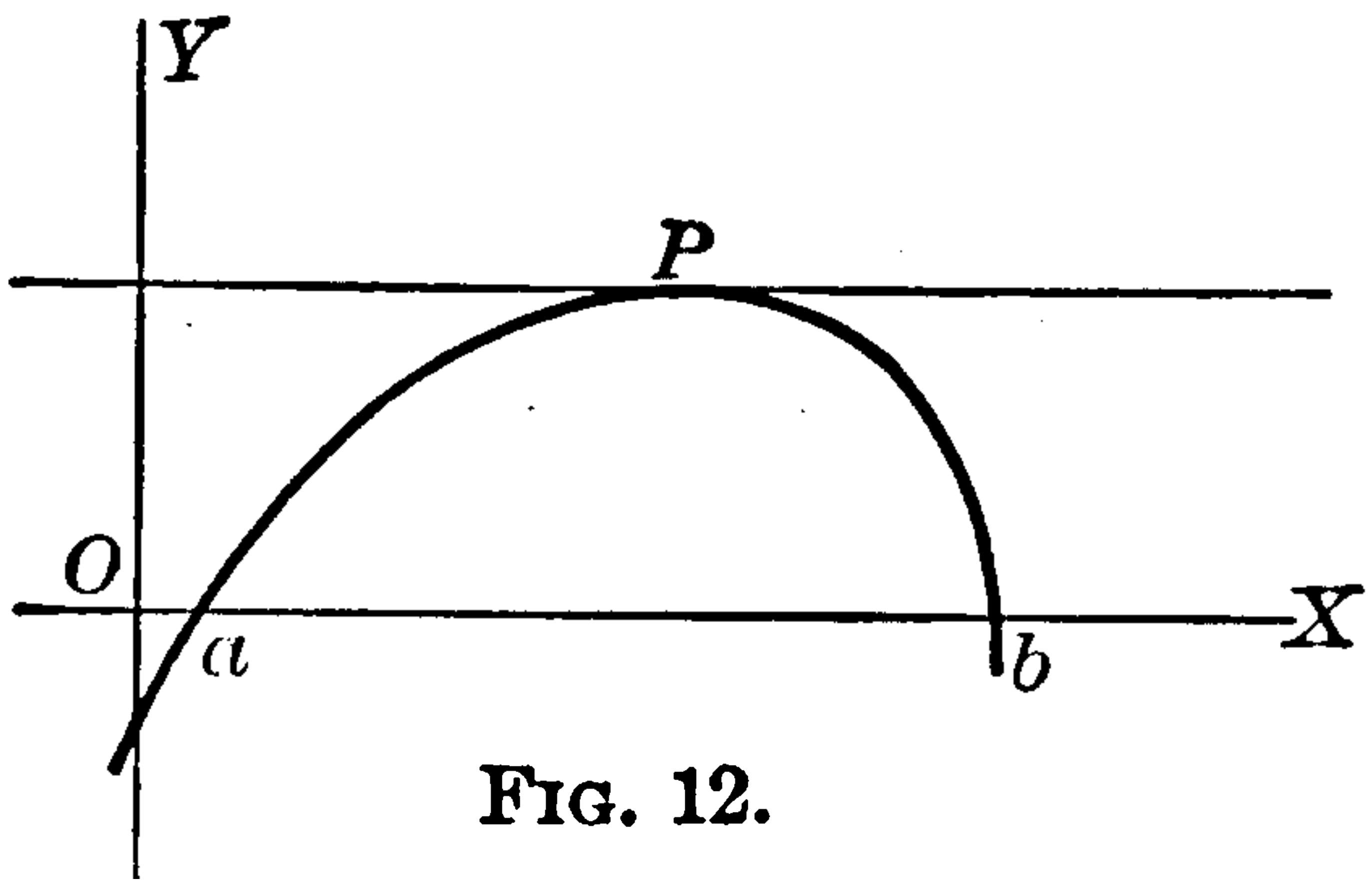


FIG. 12.

$$\frac{dy}{dx} = f'(x) = 0.$$

43. Form of remainder in Maclaurin's series. Let the remainder after n terms be denoted by $R_n(x, a)$, which is a function of x and a as well as of n. Since each of the succeeding terms is divisible by $(x-a)^n$, R_n may be conveniently written in the form

$$R_n(x, a) = \frac{(x-a)^n}{n!} \phi(x, a).$$

The problem is now to determine $\phi(x, a)$ so that the relation

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{\phi(x, a)}{n!}(x-a)^n \qquad (1)$$

may be an algebraic identity, in which the right-hand member contains only the first n terms of the series, with the remainder after n terms. Thus, by transposing,

$$f(x)-f(a)-f'(a)(x-a)-\frac{f''(a)}{2!}(x-a)^2-\cdots$$
$$-\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}-\frac{\phi(x,a)}{n!}(x-a)^n\equiv 0. \qquad (2)$$

Let a new function, $F(z)$, be defined as follows:

$$F(z)\equiv f(x)-f(z)-f'(z)(x-z)-\frac{f''(z)}{2!}(x-z)^2-\cdots$$
$$-\frac{f^{(n-1)}(z)}{(n-1)!}(x-z)^{n-1}-\frac{\phi(x,a)}{n!}(x-z)^n. \qquad (3)$$

This function $F(z)$ vanishes when $z = x$ ~~$x = z$~~, as is seen by inspection, and it also vanishes when z ~~x~~ $= a$, since it then becomes identical with the left-hand member of (2); hence, by Rolle's theorem, its derivative $F'(z)$ vanishes for some value of z between x and a, say z_1. But

$$F'(z)=-f'(z)+f'(z)-f''(z)(x-z)+f''(z)(x-z)-\cdots$$
$$-\frac{f^{(n)}(z)}{(n-1)!}(x-z)^{n-1}+\frac{\phi(x,a)}{(n-1)!}(x-z)^{n-1}.$$

These terms cancel each other in pairs except the last two; hence

$$F'(z)=\frac{(x-z)^{n-1}}{(n-1)!}[\phi(x,a)-f^{(n)}(z)].$$

Since $F'(z)$ vanishes when $z = z_1$, it follows that

$$\phi(x,a)=f^{(n)}(z_1). \qquad (4)$$

In this expression z_1 lies between x and a, and may thus be represented by

$$z_1 = a+\theta(x-a),$$

$\theta =$ [illegible]

where θ is a positive proper fraction. Hence from (4)

$$\phi(x, a) = f^{(n)}[a + \theta(x - a)],$$

and
$$R_n(x, a) = \frac{f^{(n)}[a + \theta(x - a)]}{n!}(x - a)^n.*$$

The complete form of the expansion of $f(x)$ is then

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$
$$+ \frac{f^{n-1}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(a + \theta(x - a))}{n!}(x - a)^n, \quad (5)$$

in which n is any positive integer. The series may be carried to any desired number of terms by increasing n, and the last term in (5) gives the remainder (or error) after the first n terms of the series. The symbol $f^{(n)}(a + \theta(x - a))$ indicates that $f(x)$ is to be differentiated n times with regard to x, and that x is then to be replaced by $a + \theta(x - a)$.

44. Another expression for the remainder. Instead of putting $R_n(x, a)$ in the form

$$\frac{(x - a)^n}{n!}\phi(x, a),$$

it is sometimes convenient to write it

$$R_n(x, a) = (x - a)\psi(x, a).$$

Proceeding as before, the expression for $F'(z)$ will be

$$F'(z) = -\frac{f^{(n)}(z)}{(n-1)!}(x - z)^{n-1} + \psi(x, a).$$

In order for this to vanish when $z = z_1$, it is necessary that

$$\psi(x, a) = \frac{f^{(n)}(z_1)}{(n-1)!}(x - z_1)^{n-1},$$

in which $z_1 = a + \theta(x - a)$, $x - z_1 = (x - a)(1 - \theta)$.

---

* This form of the remainder was found by Lagrange (1736–1813), who published it in the Mémoires de l'Académie des Sciences à Berlin, 1772.

Hence $\psi(x, a) = \frac{f^{(n)}(a + \theta(x-a))}{(n-1)!}(1-\theta)^{n-1}(x-a)^{n-1},$

and $R_n(x, a) = (1-\theta)^{n-1}\frac{f^{(n)}(a + \theta(x-a))}{(n-1)!}(x-a)^n.$*

An example of the use of this form of remainder is furnished by the series for $\log x$ in powers of $x - a$, when $x - a$ is negative, and also in the expansion of $(a + x)^m$.

1. Find the interval of equivalence for the development of $\log x$ in powers of $x - a$, when a is a positive number.

Here, from Art. 40, Ex. 1,

$$f^{(n)}(x)| = |\frac{(n-1)!}{x^n},$$

hence $$f^{(n)}(a + \theta(x-a))| = |\frac{(n-1)!}{(a + \theta(x-a))^n},$$

and, by Art. 43, $$R_n(x, a)| = |\frac{(x-a)^n}{n(a + \theta(x-a))^n}| = |\frac{1}{n}\left[\frac{x-a}{a+\theta(x-a)}\right]^n.$$

First let $x - a$ be positive. Then when it lies between 0 and a it is numerically less than $a + \theta(x - a)$, since θ is a positive proper fraction; hence when $n \doteq \infty$

$$\left[\frac{x-a}{a+\theta(x-a)}\right]^n \doteq 0, \text{ and } R_n(x, a) \doteq 0.$$

Again, when $x - a$ is negative and numerically less than a, the second form of the remainder must be employed. As before,

$$f^{(n)}(a + \theta(x-a))| = |\frac{(n-1)!}{(a + \theta(x-a))^n},$$

hence $$R_n(x, a)| = |(1-\theta)^{n-1} \cdot \frac{(x-a)^n}{[a + \theta(x-a)]^n}$$

$$| = |(1-\theta)^{n-1} \cdot \frac{(a-x)^n}{[a - \theta(a-x)]^n}$$

$$| = |\left[\frac{(a-x) - \theta(a-x)}{a - \theta(a-x)}\right]^{n-1} \cdot \frac{a-x}{a-\theta(a-x)}.$$

---

* This form of the remainder was found by Cauchy (1789–1857), and first published in his "Leçons sur le calcul infinitésimal," 1826.

The factor within the brackets is numerically less than 1, hence the $(n-1)$st power can be made less than any given number, by taking n large enough. This is true for all values of x between 0 and a.

Therefore, $\log x$ and its development in powers of $x-a$ are equivalent within the interval of convergence of the series, that is, for all values of x between 0 and $2a$.

Ex. **2.** Show that the development of $x^{-\frac{1}{2}}$ in positive powers of $x-a$ holds for all values of x that make the series convergent; that is, when x lies between 0 and $2a$.

If the function is expanded in powers of x, the complete form will be

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$
$$+\frac{f^{(n)}(\theta x)}{n!}x^n \qquad (1)$$

for the first form of remainder, and

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$
$$+\frac{f^{(n)}(\theta x)}{(n-1)!}(1-\theta)^{n-1}\cdot x^n \qquad (2)$$

for the second form of remainder.

Similarly, the complete form of Taylor's series (Art. 41) becomes

$$f(a+x)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}x^{n-1}$$
$$+\frac{f^{(n)}(a+\theta x)}{n!}x^n \qquad (3)$$

for the first form of remainder, and

$$f(a+x)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}x^{n-1}$$
$$+\frac{f^{(n)}(a+\theta x)}{(n-1)!}(1-\theta)^{n-1}\cdot x^n \qquad (4)$$

for the second form of remainder.

Ex. Expand $(a+x)^m$ in ascending powers of x, and determine the interval within which R_n has the limit zero.

Here
$$f(a+x)=(a+x)^m,$$
hence
$$f(x)=x^m,$$
and
$$f'(x)=mx^{m-1},\ f''(x)=m(m-1)x^{m-2},\ \ldots,$$
$$f^{(n)}(x)=m(m-1)\cdots(m-n+1)x^{m-n};$$
$$f(a)=a^m,\ f'(a)=ma^{m-1},\ f''(a)=m(m-1)a^{m-2},\ \ldots,$$
$$f^{(n)}(a)=m(m-1)\cdots(m-n+1)a^{m-n}.$$

Therefore
$$(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{2\,!}a^{m-2}x^2+\cdots$$
$$+\frac{m(m-1)\cdots(m-n+2)}{(n-1)\,!}a^{m-n+1}x^{n-1}+R_n(x,a),$$
in which, from the first form of remainder,
$$R_n(x,a)=\frac{m(m-1)\cdots(m-n+1)}{n\,!}(a+\theta x)^{m-n}x^n$$
$$=\frac{m(m-1)\cdots(m-n+1)}{n\,!}(a+\theta x)^m\left(\frac{x}{a+\theta x}\right)^n.$$

Consider the ratio
$$\frac{R_n}{R_{n-1}}=\frac{m-n+1}{n}\cdot\frac{x}{a+\theta x}.$$

When m is greater than -1, the factor $\dfrac{m-n+1}{n}$ is less than unity for every value of n greater than $\tfrac{1}{2}(m+1)$, and when x lies between 0 and a the second factor is also less than unity. Their product is therefore less than some fixed proper fraction k for every such value of n.

Hence
$$R_n<kR_{n-1}<k^2R_{n-2}\cdots<k^{n-t}R_t. \qquad [t>\tfrac{1}{2}(m+1).$$

Since R_t is finite and k^{n-t} can be made as small as desired by taking n sufficiently large, it follows that
$$\lim_{n\doteq\infty}R_n(x,a)=0$$
when
$$m>-1,\ \text{and}\ 0\leq x<a.$$

When m is not greater than -1, let it be replaced by $-p$, where p is equal to or greater than 1. The ratio R_n to R_{n-1} may now be written
$$(-1)\left(\frac{p-1}{n}+1\right)\cdot\frac{x}{a+\theta x}.$$

The factor $\left(\frac{p-1}{n}+1\right)$ is equal to or greater than unity, but in the latter case becomes more and more nearly unity as n increases.

The factor $\frac{x}{a+\theta x}$ is numerically less than 1 for any value of n when x is given any value between zero and a. The product is therefore greater than unity for values of n less than some finite number N, but less than unity for values of n greater than N. Then, in the same way as for the preceding case,

$$R_{N+1} < kR_N,$$

$$R_{N+2} < kR_{N+1} < k^2R_N,$$

in which R_N is a finite number and k a proper fraction. Hence

$$\lim_{n \doteq \infty} R_n(x, a) = 0$$

for every value of m, provided x lies between 0 and a.

Since the interval of convergence is, by Art. 38, from $x = -a$ to $x = a$, it remains to examine the value of $R_n(x, a)$ when x lies between 0 and $-a$. For this purpose it is necessary to use the second form of remainder, which may be written,

$$R_n(x, a) = \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} \cdot x(a+\theta x)^{m-1}\left(\frac{1-\theta}{1+\frac{\theta x}{a}}\right)^{n-1} \cdot \left(\frac{x}{a}\right)^{n-1}.$$

But when $\frac{x}{a}$ is negative and less than 1, the expression $\frac{1-\theta}{1+\theta\frac{x}{a}}$ is a proper fraction, hence its $(n-1)$st power approaches zero as a limit. The expression $\frac{(m-1)\ldots(m-n+1)}{(n-1)!}\left(\frac{x}{a}\right)^{n-1}$ is made up of $n-1$ factors of the form $\frac{m-k}{k} \cdot \frac{x}{a}$, each of which is finite and, when k is sufficiently large, is numerically less than 1; hence the limit of the entire product is zero. Therefore

$$R_n(x, a) \doteq 0, \text{ when } n \doteq \infty,$$

if x lies between $-a$ and $+a$.

Therefore, for values of x within this interval the function $(a+x)^m$ and its expansion are equivalent.

45. Theorem of mean value. Let $f(x)$ be a continuous function of x which has a derivative. It can then be represented by the ordinates of a curve whose equation is $y = f(x)$.

In Fig. 13, let

$$x = ON,\ x + h = OR,$$

$$f(x) = NH,\ f(x+h) = RK.$$

Then $f(x+h) - f(x) = MK$, and

$$\frac{f(x+h)-f(x)}{h} = \frac{MK}{HM} = \tan MHK.$$

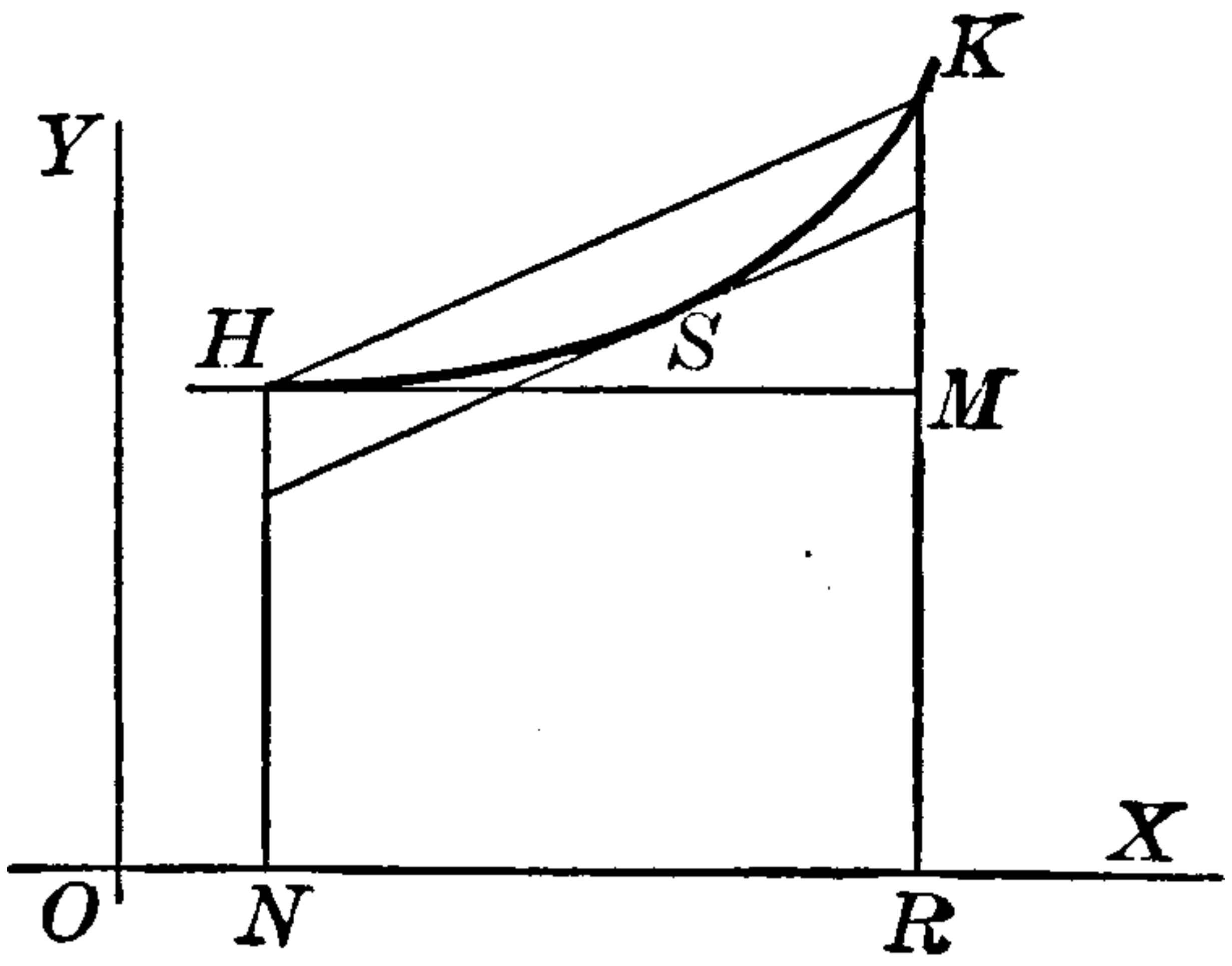


FIG. 13.

But at some point S between H and K the tangent to the curve is parallel to the secant HK. Since the abscissa of S is greater than x and less than $x + h$ it may be represented by $x + \theta h$, in which θ is a positive number less than unity. The slope of the tangent at S is then expressed by $f'(x+\theta h)$, hence

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h),$$

from which

$$f(x+h) = f(x) + hf'(x+\theta h).$$

The theorem expressed by this formula is known as the *theorem of mean value.*

The theorem of mean value can also be established from Taylor's theorem by putting $n = 1$ in equation (3) of Art. 44.

1 here

EXERCISES ON CHAPTER IV

1. Expand $\cos(x+h)$ in powers of h.

2. Expand $\tan(x+h)$ in powers of x.

3. By expanding $\cos(x+h)$ in powers of one of its variables, prove the theorem $\cos(x+h) = \cos x \cos h - \sin x \sin h$.

4. Expand $\log(x+h)$ by Taylor's theorem in powers of h.

5. Expand $(x+y)^5$ in powers of y.

6. Prove $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{2^3} - \frac{x^4}{2^3 \cdot 4!} + R.$

7. Prove $\log\frac{x}{x-1} = \frac{1}{x-1} - \frac{1}{2(x-1)^2} + \frac{1}{3(x-1)^3} + R.$

8. Prove $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + R.$

9. If $f(x) = f(-x)$, prove that the expansion of $f(x)$ in powers of x will contain only even powers of x, as $\cos x$, $\sqrt{1-x^2}$; if $f(x) = -f(-x)$, the expansion of $f(x)$ will involve only odd powers of x, as $\sin x$, $\tan x$, $\frac{x}{1+x^2}$.

10. Show that in the expansion for $\sin x$ in powers of x,

$$\lim_{n \doteq \infty} R_n(x) = 0.$$

$\frac{1}{x}$

$\log\frac{x}{x-1}$ $\log\left(1+\frac{1}{x-1}\right)$

$f(a+x) = \log(1+u)_{u=\frac{1}{x-1}}$

$f(a) = \log a\Big|_{a=1} = 0$

$f'(a) = \frac{1}{a}\Big|_{a=1} = 1$

$f''(a) = -\frac{1}{a^2} = -1$

$f'''(a) = \frac{2}{a^3} = 2$

$f(x+a) = f(a) + f'(a)x + f''(a)\frac{x^2}{\lfloor 2} + f'''(a)\frac{x^3}{\lfloor 3}$

$= 0 + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + \frac{2}{\lfloor 3 (x-1)^3}$

CHAPTER V

INDETERMINATE FORMS

46. Hitherto the values of a given function $f(x)$, corresponding to assigned values of the variable x, have been obtained by direct substitution. The function may, however, involve the variable in such a way that for certain values of the latter the corresponding values of the function cannot be found by mere substitution.

For example, the function

$$\frac{e^x - e^{-x}}{\sin x},$$

for the value $x=0$, assumes the form $\frac{0}{0}$, and the corresponding value of the function is thus not directly determined. In such a case the expression for the function is said to assume an *indeterminate form* for the assigned value of the variable.

The example just given illustrates the indeterminateness of most frequent occurrence; namely, that in which the given function is the quotient of two other functions that vanish for the same value of the variable.

Thus if $$f(x)=\frac{\phi(x)}{\psi(x)},$$

and if, when x takes the special value a, the functions $\phi(x)$ and $\psi(x)$ both vanish, then

$$f(a)=\frac{\phi(a)}{\psi(a)}=\frac{0}{0}$$

is indeterminate in form, and cannot be rendered determinate without further transformation.

47. Indeterminate forms may have determinate values. A case has already been noticed (Art. 9) in which an expression that assumes the form $\frac{0}{0}$ for a certain value of its variable takes a definite value, dependent upon the law of variation of the function in the vicinity of the assigned value of the variable.

As another example, consider the function

$$y = \frac{x^2 - a^2}{x - a}.$$

If this relation between x and y be written in the forms

$$y(x - a) = x^2 - a^2, \quad (x - a)(y - x - a) = 0,$$

it will be seen that it can be represented graphically, as in the figure (Fig. 14), by the pair of lines

$$x - a = 0,$$

$$y - x - a = 0.$$

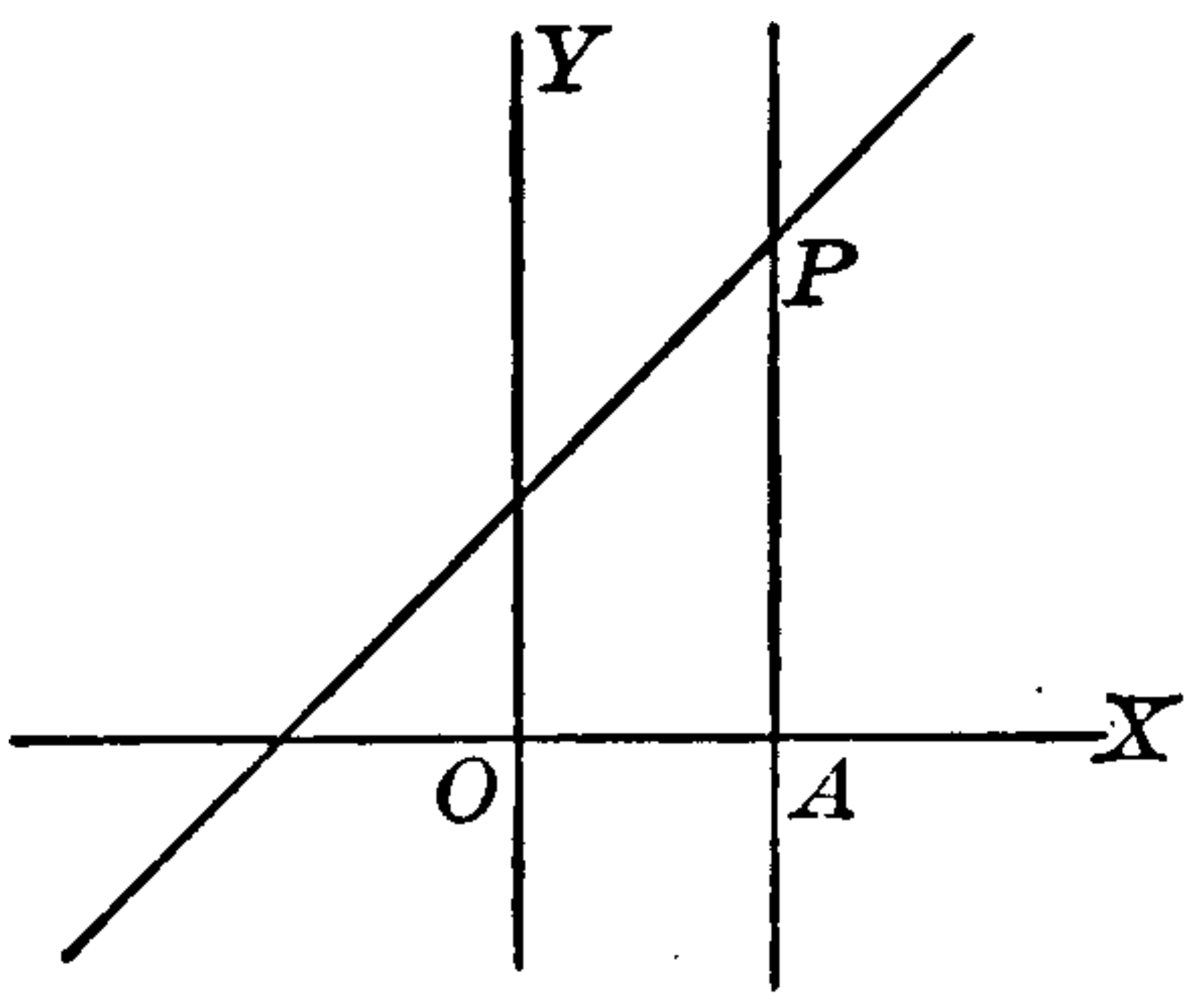


Fig. 14.

Hence when x has the value of a there is an indefinite number of corresponding points on the locus, all situated on the line $x = a$; and accordingly for this value of x the function y may have any value whatever, and is therefore indeterminate.

When x has any value different from a, the corresponding value of y is determined from the equation $y = x + a$. Now, of the infinite number of different values of y corresponding to $x = a$, there is one particular value AP which is continuous with the series of values taken by y when x takes successive values in the vicinity of $x = a$. This may be called the *determinate value* of y when $x = a$. It is obtained by putting $x = a$ in the equation $y = x + a$, and is therefore $y = 2a$.

This result may be stated without reference to a locus as follows: When $x = a$, the function

$$\frac{x^2 - a^2}{x - a}$$

is indeterminate, and has an infinite number of different values; but among these values there is one determinate value which is continuous with the series of values taken by the function as x increases through the value a; this determinate or singular value may then be defined by

$$\lim_{x \doteq a} \frac{x^2 - a^2}{x - a}.$$

In evaluating this limit the infinitesimal factor $x - a$ may be removed from numerator and denominator, since this factor is not zero, while x is different from a; hence the determinate value of the function is

$$\lim_{x \doteq a} \frac{x + a}{1} = 2a.$$

Ex. 1. Find the determinate value, when $x \doteq 1$, of the function

$$\frac{x^3 + 2x^2 - 3x}{3x^3 - 3x^2 - x + 1},$$

which, at the limit, takes the form $\frac{0}{0}$.

This expression may be written in the form

$$\frac{(x^2 + 3x)(x - 1)}{(3x^2 - 1)(x - 1)},$$

which reduces to $\frac{x^2 + 3x}{3x^2 - 1}$. When $x = 1$, this becomes $\frac{4}{2} = 2$.

Ex. **2.** Evaluate the expression

$$\frac{x^3 + ax^2 + a^2x + a^3}{x^3 + xb^2 + ax^2 + ab^2}$$

when $x \doteq -a$.

Ex. 3. Determine the value of

$$\frac{x^3-7x^2+3x+14}{x^3+3x^2-17x+14}$$

when $x \doteq 2$.

Ex. 4. Evaluate $\dfrac{a-\sqrt{a^2-x^2}}{x^2}$ when $x \doteq 0$.

(Multiply both numerator and denominator by $a+\sqrt{a^2-x^2}$.)

48. Evaluation by development. In some cases the common vanishing factor can be best removed after expansion in series.

Ex. 1. Consider the function mentioned in Art. 46,

$$\frac{e^x-e^{-x}}{\sin x}.$$

When numerator and denominator are developed in powers of x, the expression becomes

$$\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)}{x-\frac{x^3}{3!}+\cdots}$$

$$=\frac{2x+\frac{2}{3!}x^3+\cdots}{x-\frac{x^3}{3!}+\cdots}=\frac{2+\frac{x^2}{3}+\cdots}{1-\frac{x^2}{6}+\cdots},$$

which has the determinate value 2, when x takes the value zero.

Ex. 2. As another example, evaluate, when $x \doteq 0$, the function

$$\frac{x-\sin^{-1}x}{\sin^3 x}.$$

By development it becomes

$$\frac{x-\left(x+\frac{1}{2}\cdot\frac{x^3}{3}+\cdots\right)}{\left(x-\frac{x^3}{3!}+\cdots\right)^3}=\frac{\frac{x^3}{6}+\cdots}{x^3+\cdots}.$$

Removing the common factor, and then putting $x = 0$, the result is $\frac{1}{6}$.

In these two examples the assigned value of x, for which the indeterminateness occurs, is zero, and the developments are made in powers of x. If the assigned value of x be some other number, as a, then the development should be made in powers of $x-a$.

Ex. 3. Evaluate, when $x=\frac{\pi}{2}$, the function

$$\frac{\cos x}{1-\sin x}.$$

By putting $x-\frac{\pi}{2}=h$, $x=\frac{\pi}{2}+h$, the expression becomes

$$\frac{\cos\left(\frac{\pi}{2}+h\right)}{1-\sin\left(\frac{\pi}{2}+h\right)}=\frac{-\sin h}{1-\cos h}=\frac{-h+\frac{h^3}{6}-\cdots}{\frac{h^2}{2}-\frac{h^4}{24}+\cdots}=\frac{-1+\frac{h^2}{6}-\cdots}{\frac{h}{2}-\frac{h^3}{24}+\cdots},$$

which becomes infinite when $h=0$, that is, when $x=\frac{\pi}{2}$.

Hence $$\lim_{x\doteq\frac{\pi}{2}}\frac{\cos x}{1-\sin x}=\pm\infty,$$

according as h approaches zero from the negative or positive side.

49. Evaluation by differentiation. Let the given function be of the form $\frac{f(x)}{\phi(x)}$, and suppose that $f(a)=0$, $\phi(a)=0$. It is required to find $\lim\limits_{x\doteq a}\frac{f(x)}{\phi(x)}$.

As before, let $f(x)$, $\phi(x)$ be developed in the vicinity of $x=a$, by expanding them in powers of $x-a$. Then

$$\frac{f(x)}{\phi(x)}=\frac{f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots}{\phi(a)+\phi'(a)(x-a)+\frac{\phi''(a)}{2!}(x-a)^2+\cdots}$$

$$=\frac{f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots}{\phi'(a)(x-a)+\frac{\phi''(a)}{2!}(x-a)^2+\cdots}.$$

By dividing by $x-a$ and then letting $x \doteq a$, it follows that

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}.$$

The functions $f'(a)$, $\phi'(a)$ will in general both be finite.

If $f'(a) = 0$, $\phi'(a) \neq 0$, then $\frac{f(a)}{\phi(a)} = 0$.

If $f'(a) \neq 0$, $\phi'(a) = 0$, then $\frac{f(a)}{\phi(a)} = \infty$.

If $f'(a)$ and $\phi'(a)$ are both zero, the limiting value of $\frac{f(x)}{\phi(x)}$ is to be obtained by carrying Taylor's development one term farther, removing the common factor $(x-a)^2$, and then letting x approach a. The result is $\frac{f''(a)}{\phi''(a)}$.

Similarly, if $f(a)$, $f'(a)$, $f''(a)$; $\phi(a)$, $\phi'(a)$, $\phi''(a)$ all vanish, it is proved in the same manner that

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'''(a)}{\phi'''(a)},$$

and so on, until a result is obtained that is not indeterminate in form.

Hence the rule:

To evaluate an expression of the form $\frac{0}{0}$, *differentiate numerator and denominator separately; substitute the critical value of x in their derivatives, and equate the quotient of the derivatives to the indeterminate form.*

Ex. **1.** Evaluate $\frac{1-\cos\theta}{\theta^2}$ when $\theta = 0$.

Put $f(\theta) = 1 - \cos\theta$, $\phi(\theta) = \theta^2$.

Then $f'(\theta) = \sin\theta$, $\phi'(\theta) = 2\theta$,

and $f'(0) = 0$, $\phi'(0) = 0$.

Again, $$f''(\theta) = \cos\theta, \qquad \phi''(\theta) = 2,$$

$$f''(0) = 1, \qquad \phi''(0) = 2,$$

hence $$\lim_{\theta \doteq 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}.$$

Ex. **2**. Find $\lim\limits_{x \doteq 0} \dfrac{e^x + e^{-x} + 2\cos x - 4}{x^4}.$

$$\begin{aligned}\lim_{x \doteq 0} \frac{e^x + e^{-x} + 2\cos x - 4}{x^4} &= \lim_{x \doteq 0} \frac{e^x - e^{-x} - 2\sin x}{4x^3} \\ &= \lim_{x \doteq 0} \frac{e^x + e^{-x} - 2\cos x}{12x^2} \\ &= \lim_{x \doteq 0} \frac{e^x - e^{-x} + 2\sin x}{24x} \\ &= \lim_{x \doteq 0} \frac{e^x + e^{-x} + 2\cos x}{24} \\ &= \frac{1}{6}.\end{aligned}$$

Ex. **3**. Find $\lim\limits_{x \doteq 0} \dfrac{x - \sin x \cos x}{x^3}.$

Ex. **4**. Find $\lim\limits_{x \doteq 1} \dfrac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}.$

In this example, show that $x - 1$ is a factor of both numerator and denominator.

Ex. **5**. Find $\lim\limits_{x \doteq 0} \dfrac{3\tan x - 3x - x^3}{x^5}.$

In applying this process to particular problems, the work can often be shortened by evaluating a non-vanishing factor in either numerator or denominator before performing the differentiation.

Ex. **6**. Find $\lim\limits_{x \doteq 0} \dfrac{(x-4)^2 \tan x}{x}.$

The given expression may be written

$$\lim_{x \doteq 0} (x-4)^2 \frac{\tan x}{x} = \lim_{x \doteq 0} (x-4)^2 \lim_{x \doteq 0} \frac{\tan x}{x}$$

$$= 16 \cdot 1 = 16.$$

In general, if $f(x) = \psi(x)\chi(x)$, and if $\psi(a) = 0$, $\chi(a) \neq 0$, $\phi(a) = 0$, then

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \chi(a)\frac{\psi'(a)}{\phi'(a)}.$$

For

$$\lim_{x \doteq a} \frac{\psi(x)\chi(x)}{\phi(x)} = \lim_{x \doteq a} \chi(x) \cdot \lim_{x \doteq a} \frac{\psi(x)}{\phi(x)} = \chi(a) \cdot \frac{\psi'(a)}{\phi'(a)}.$$

Ex. **7.** Find $\lim\limits_{x \doteq \frac{\pi}{2}} \dfrac{\sin x \cos^2 x}{(2x - \pi)^2}$.

Ex. **8.** Find $\lim\limits_{x \doteq 1} \dfrac{(x-3)^2 \log(2-x)}{\sin(x-1)}$.

EXERCISES

Evaluate the following expressions:

9. $\dfrac{1 - \cos x}{\sin x}$ when $x = 0$.

10. $\dfrac{e^x - e^{-x}}{\sin x}$ when $x = 0$.

11. $\dfrac{x^3 - 1}{x - 1}$ when $x = 1$.

12. $\dfrac{a^x - 1}{b^x - 1}$ when $x = 0$.

13. $\dfrac{\sin ax}{\sin bx}$ when $x = 0$.

14. $\dfrac{(1+x)^{\frac{1}{n}} - 1}{x}$ when $x = 0$.

15. $\dfrac{e^x + e^{-x} - 2}{x^2}$ when $x = 0$.

16. $\dfrac{\tan x - \sin x \cos x}{x^3}$ when $x = 0$.

17. $\dfrac{\sin^{-1} x}{\tan^{-1} x}$ when $x = 0$.

18. $\dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$ when $x = 0$.

19. $\dfrac{\tan x - \sin x}{x^3}$ when $x = 0$.

There are other indeterminate forms than $\frac{0}{0}$. They are $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^∞, ∞^0.

50. Evaluation of the indeterminate form $\frac{\infty}{\infty}$.

Let the function $\dfrac{f(x)}{\phi(x)}$ become $\dfrac{\infty}{\infty}$ when $x = a$. It is required to find $\lim\limits_{x \doteq a} \dfrac{f(x)}{\phi(x)}$.

This function can be written

$$\frac{f(x)}{\phi(x)}=\frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}},$$

which takes the form $\frac{0}{0}$ when $x = a$, and can therefore be evaluated by the preceding rule.

When $x \doteq a$,

$$\lim_{x \doteq a}\frac{f(x)}{\phi(x)}=\lim_{x \doteq a}\frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}}=\lim_{x \doteq a}\frac{-\dfrac{\phi'(x)}{[\phi(x)]^2}}{-\dfrac{f'(x)}{[f(x)]^2}}$$

$$=\lim_{x \doteq a}\left[\frac{f(x)}{\phi(x)}\right]^2\frac{\phi'(x)}{f'(x)}. \qquad (1)$$

If both members be divided by $\frac{f(x)}{\phi(x)}$, the equation becomes

$$1=\lim_{x \doteq a}\frac{f(x)}{\phi(x)}\,\frac{\phi'(x)}{f'(x)},$$

therefore

$$\lim_{x \doteq a}\left[\frac{f(x)}{\phi(x)}\right]=\frac{f'(a)}{\phi'(a)}. \qquad (2)$$

This is exactly the same result as was obtained for the form $\frac{0}{0}$; hence the procedure for evaluating the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, is the same in both cases.

When the true value of $\frac{f(a)}{\phi(a)}$ is 0 or ∞, equation (1) is satisfied, independent of the value of $\frac{f'(a)}{\phi'(a)}$; but (2) still gives the correct value. For, suppose $\lim_{x \doteq a}\frac{f(x)}{\phi(x)}=0$. Consider the function

$$\frac{f(x)}{\phi(x)}+c=\frac{f(x)+c\phi(x)}{\phi(x)},$$

which has the form $\frac{\infty}{\infty}$ when $x = a$, and has the determinate value c, which is not zero. Hence by (2)

$$\lim_{x \doteq a} \frac{f(x) + c\phi(x)}{\phi(x)} = \frac{f'(a) + c\phi'(a)}{\phi'(a)} = \frac{f'(a)}{\phi'(a)} + c.$$

Therefore, by subtracting c,

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}.$$

If $\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \infty$, then $\lim_{x \doteq a} \frac{\phi(x)}{f(x)} = 0$, which can be treated as the previous case.

51. Evaluation of the form $\infty \cdot 0$.

Let the function be $\phi(x) \cdot \psi(x)$, such that $\phi(a) = \infty$, $\psi(a) = 0$.

This may be written $\dfrac{\psi(x)}{\dfrac{1}{\phi(x)}}$, which takes the form $\frac{0}{0}$ when a is substituted for x, and therefore comes under the above rule. (Art. 49.)

52. Evaluation of the form $\infty - \infty$.

The form $\infty - \infty$ may be finite, zero, or infinite.

For instance, consider $\sqrt{x^2 + ax} - x$ for the value $x = \infty$. It is of the form $\infty - \infty$, but by multiplying and dividing by $\sqrt{x^2 + ax} + x$ it becomes $\dfrac{ax}{\sqrt{x^2 + ax} + x}$, which has the form $\frac{\infty}{\infty}$ when $x = \infty$.

Again, by dividing both terms by x, it takes the form $\dfrac{a}{\sqrt{1 + \dfrac{a}{x}} + 1}$, which becomes $\frac{a}{2}$ when $x = \infty$.

There is here no general rule of procedure as in the previous cases, but by means of transformations and proper grouping of terms it is often possible to bring it into one of the forms $\frac{0}{0}, \frac{\infty}{\infty}$. Frequently a function which becomes $\infty - \infty$ for a critical value of x can be put in the form

$$\frac{u}{v} - \frac{t}{w},$$

in which v, w become zero. This can be reduced to

$$\frac{uw - vt}{vw},$$

which is then of the form $\frac{0}{0}$.

Ex. 1. Find $\underset{x \doteq \frac{\pi}{2}}{\lim} (\sec x - \tan x)$.

This expression assumes the form $\infty - \infty$, but can be written

$$\frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x},$$

which is of the form $\frac{0}{0}$, and gives zero when evaluated.

Hence $\underset{x \doteq \frac{\pi}{2}}{\lim} (\sec x - \tan x) = 0.$

Ex. 2. Prove $\underset{x \doteq \frac{\pi}{2}}{\lim} (\sec^n x - \tan^n x) = \infty, 1, 0$, according as

$$n > 2, \quad = 2, \quad < 2.$$

where

$\frac{\sin x}{x}$ when

EXERCISES

Evaluate the following expressions:

1. $\frac{\log \sin 2x}{\log \sin x}$ when $x = 0$.
2. $\frac{\log x}{\cot x}$ when $x = 0$.
3. $\frac{x^n}{e^x}$ when $x = \infty$.
4. $\frac{\tan x}{\tan 5x}$ when $x = \frac{\pi}{2}$.
5. $(a^{\frac{1}{x}} - 1)x$ when $x = \infty$.
6. $\frac{\sec 3x}{\sec 5x}$ when $x = \frac{\pi}{2}$.
7. $(a^2 - x^2) \tan \frac{\pi x}{2a}$ when $x = a$.
8. $(1 - \tan x) \sec 2x$ when $x = \frac{\pi}{4}$.

9. $\dfrac{1-\log x}{e^{\frac{1}{x}}}$ when $x=0$.

10. $\dfrac{\log x}{x^n}$ when $x=\infty$.

11. $\dfrac{1}{\log x}-\dfrac{1}{x-1}$ when $x=1$.

12. $\dfrac{e}{e^x-e}-\dfrac{1}{x-1}$ when $x=1$.

13. $\csc^2 x-\dfrac{1}{x^2}$ when $x=0$.

14. $\dfrac{1}{\log x}-\dfrac{x}{\log x}$ when $x=1$.

53. Evaluation of the form 1^∞.

Let the function $u=[\phi(x)]^{\psi(x)}$ assume the form 1^∞ when $x=a$.

In order to evaluate this expression, take the logarithm of both sides. Then

$$\log u=\psi(x)\cdot\log\phi(x)=\frac{\log\phi(x)}{\dfrac{1}{\psi(x)}}.$$

This expression assumes the form $\dfrac{0}{0}$ when $x=a$, and can be evaluated by the method of Art. 49.

If the reduced value of this fraction be denoted by m, then $\log u=m$ and $u=e^m$.

NOTE. The form 1^0 is not indeterminate, but is equal to 1. For, let $[\phi(x)]^{\psi(x)}$ assume the form 1^0 when $x=a$.

Put $$u=[\phi(x)]^{\psi(x)}.$$

Then $$\log u=\psi(x)\log[\phi(x)],$$

which equals zero when $x=a$;

hence $$\log u=0,\quad u=e^0=1.$$

54. Evaluation of the forms ∞^0, 0^0.

Let $[\phi(x)]^{\psi(x)}$ become ∞^0 when $x=a$.

Put $$u=[\phi(x)]^{\psi(x)}.$$

Then $$\log u=\psi(x)\log\phi(x)=\frac{\log\phi(x)}{\dfrac{1}{\psi(x)}}.$$

This is of the form $\frac{\infty}{\infty}$, and can be evaluated by the method of Art. 50. Similarly for the form 0^0.

NOTE. The form 0^∞ is not indeterminate, but is equal to 0.

For let $u = [\phi(x)]^{\psi(x)}$ become 0^∞ when $x = a$.

Then $\log u = \psi(x) \log \phi(x) = -\infty$, and $u = e^{-\infty} = 0$.

Similarly, the form $0^{-\infty}$ is equal to ∞.

This completes the list of ordinary indeterminate forms. The evaluation of all of them depends upon the same principle, namely, that each form (or its logarithm) may be brought to the form $\frac{0}{0}$, and then evaluated by differentiating numerator and denominator separately. In finally letting $x \doteq a$, the two directions of approach should be compared, so as to reveal any discontinuity in the function.

EXERCISES

Evaluate the following indeterminate forms:

1. $x^{\sin x}$ when $x = 0$.

2. $(\cos ax)^{\csc^2 cx}$ when $x = 0$.

3. $(1 + ax)^{\frac{b}{x}}$ when $x = 0$.

4. $(1 + ax)^{\frac{b}{x}}$ when $x = \infty$.

5. $\left(\frac{1}{x}\right)^{\sin x}$ when $x = 0$.

6. $\left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$ when $x = a$.

7. x^x when $x = 0$.

8. $x^{\frac{1}{x}}$ when $x = \infty$.

9. $\left(\frac{a}{x} + 1\right)^x$ when $x = \infty$.

EXERCISES ON CHAPTER V

Evaluate the following indeterminate forms:

✓ **1.** $\frac{e^x - 1}{x}$ when $x = 0$.

✓ **2.** $\frac{x^2}{\sin x}$ when $x = 0$.

✓ **3.** $\frac{e^{mx} - e^{ma}}{(x - a)^r}$ when $x = a$.

✓ **4.** $\frac{\sqrt{x} \tan x}{\sqrt{(e^x - 1)^3}}$ when $x = 0$.

✓ **5.** $\frac{\log x}{\csc x}$ when $x = 0$.

✓ **6.** $e^{-x} \log x$ when $x = \infty$.

✓ **7.** $x - \sqrt{x^2 - ax}$ when $x = \infty$.

8. $(\cot x)^{\sin x}$ when $x = 0$.

✓ **9.** $\left[\dfrac{a_1^{nx} + a_2^{nx} + \cdots a_n^{nx}}{n}\right]^{\frac{1}{x}}$ when $x = 0$.

10. $\dfrac{1}{x} - \cot x$ when $x = 0$.

11. $\dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1}$ when $x = 1$.

✓ **12.** $2^x \sin \dfrac{a}{2^x}$ when $x = \infty$.

13. $\dfrac{\sqrt{x} - \sqrt{b} + \sqrt{x - b}}{\sqrt{x^2 - b^2}}$ when $x = b$.

14. $\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$ when $x = 0$.

15. $\dfrac{\sqrt{2} - \sin x - \cos x}{\log \sin 2x}$ when $x = \dfrac{\pi}{4}$.

✓ **16.** $x \tan x - \dfrac{\pi}{2} \sec x$ when $x = \dfrac{\pi}{2}$.

17. $\left(\dfrac{\log x}{x}\right)^{\frac{1}{x}}$ when $x = \infty$.

✓ **18.** $x - x^2 \log\left(1 + \dfrac{1}{x}\right)$ when $x = \infty$.

$\dfrac{\tan x}{x} = 1$ $x \to 0$

CHAPTER VI

MODE OF VARIATION OF FUNCTIONS OF ONE VARIABLE

55. In this chapter methods of exhibiting the march or mode of variation of functions, as the variable takes all values in succession from $-\infty$ to $+\infty$, will be discussed. Simple examples have been given in Art. 13 of the use that can be made of the derivative function $\phi'(x)$ for this purpose.

The fundamental principle employed is that when x increases through the value a, $\phi(x)$ increases through the value $\phi(a)$ if $\phi'(a)$ is positive, and that $\phi(x)$ decreases through the value $\phi(a)$ if $\phi'(a)$ is negative. Thus the question of finding whether $\phi(x)$ increases or decreases through an assigned value $\phi(a)$, is reduced to determining the sign of $\phi'(a)$.

1. Find whether the function

$$\phi(x) \equiv x^2 - 4x + 5$$

increases or decreases through the values $\phi(3)=2$, $\phi(0)=5$, $\phi(2)=1$, $\phi(-1)=10$, and state at what value of x the function ceases to increase and begins to decrease, or conversely.

56. Turning values of a function. It follows that the values of x at which $\phi(x)$ ceases to increase and begins to decrease are those at which $\phi'(x)$ changes sign from positive to negative; and that the values of x at which $\phi(x)$ ceases to decrease and begins to increase are those at which $\phi'(x)$ changes its sign from negative to positive. In the former case, $\phi(x)$ is said to pass through a *maximum*, in the latter, a *minimum* value.

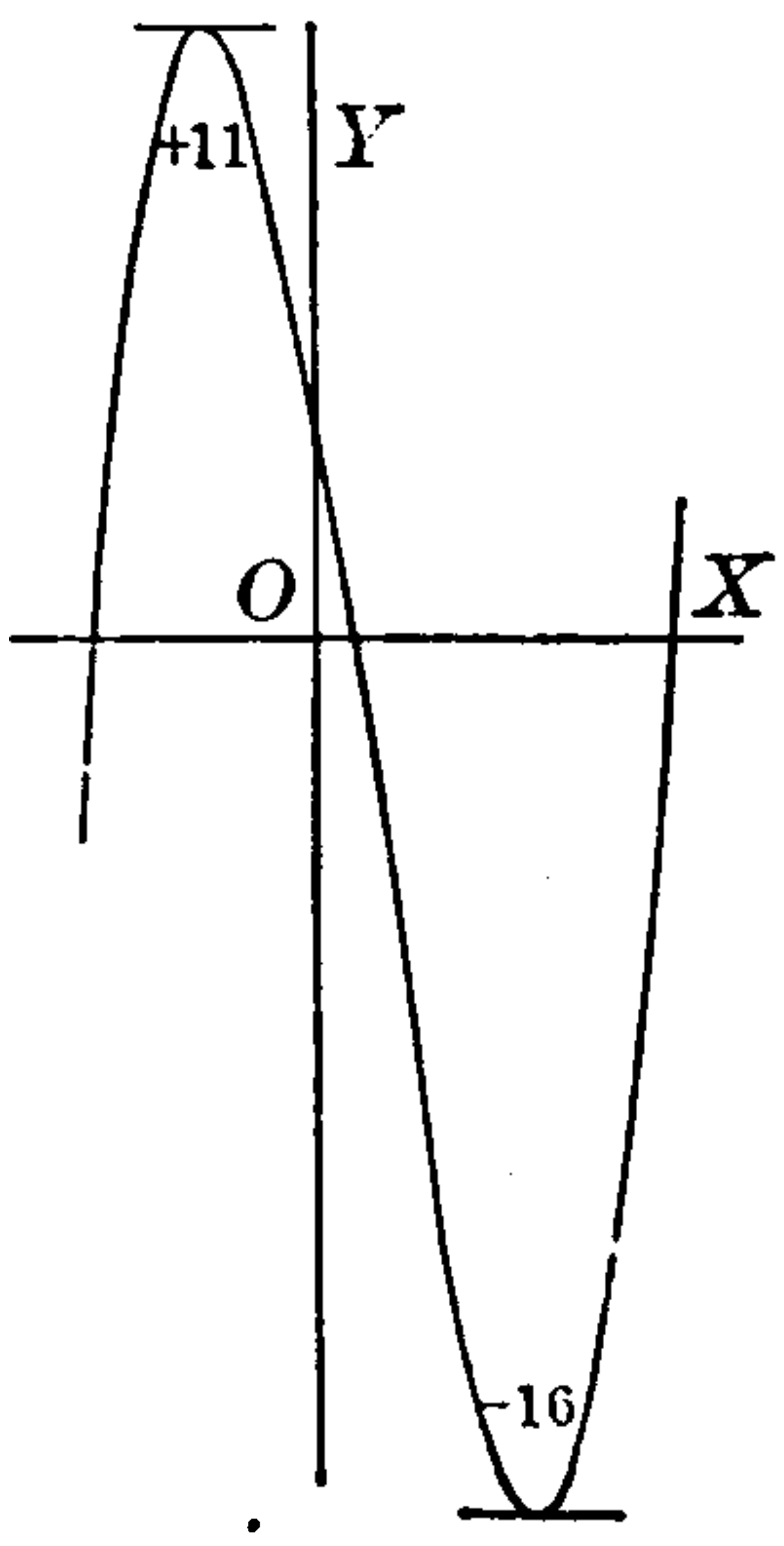


FIG. 15.

Ex. **1.** Find the turning values of the function

$$\phi(x) = 2x^3 - 3x^2 - 12x + 4,$$

and exhibit the mode of variation of the function by sketching the curve $y = \phi(x)$.

Here $\phi'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$,

hence $\phi'(x)$ is negative when x lies between -1 and $+2$, and positive for all other values of x. Thus $\phi(x)$ increases from $x = -\infty$ to $x = -1$, decreases from $x = -1$ to $x = 2$, and increases from $x = 2$ to $x = \infty$. Hence $\phi(-1)$ is a maximum value of $\phi(x)$, and $\phi(2)$ a minimum.

The general form of the curve $y = \phi(x)$ (Fig. 15) may be inferred from the last statement, and from the following simultaneous values of x and y:

$x =$	$-\infty$,	-2,	-1,	0,	1,	2,	3,	4,	∞.
$y =$	$-\infty$,	0,	11,	4,	-9,	-16,	-5,	36,	∞.

Ex. **2.** Exhibit the variation of the function

$$\phi(x) = (x-1)^{\frac{2}{3}} + 2,$$

especially its turning values.

Since $\phi'(x) = \dfrac{2}{3}\dfrac{1}{(x-1)^{\frac{1}{3}}}$,

hence $\phi'(x)$ changes sign at $x = 1$, being negative when $x < 1$, infinite when $x = 1$, and positive when $x > 1$. Thus $\phi(1) = 2$ is a minimum turning value of $\phi(x)$. The graph of the function is as shown in Fig. 16, with a vertical tangent at the point (1, 2).

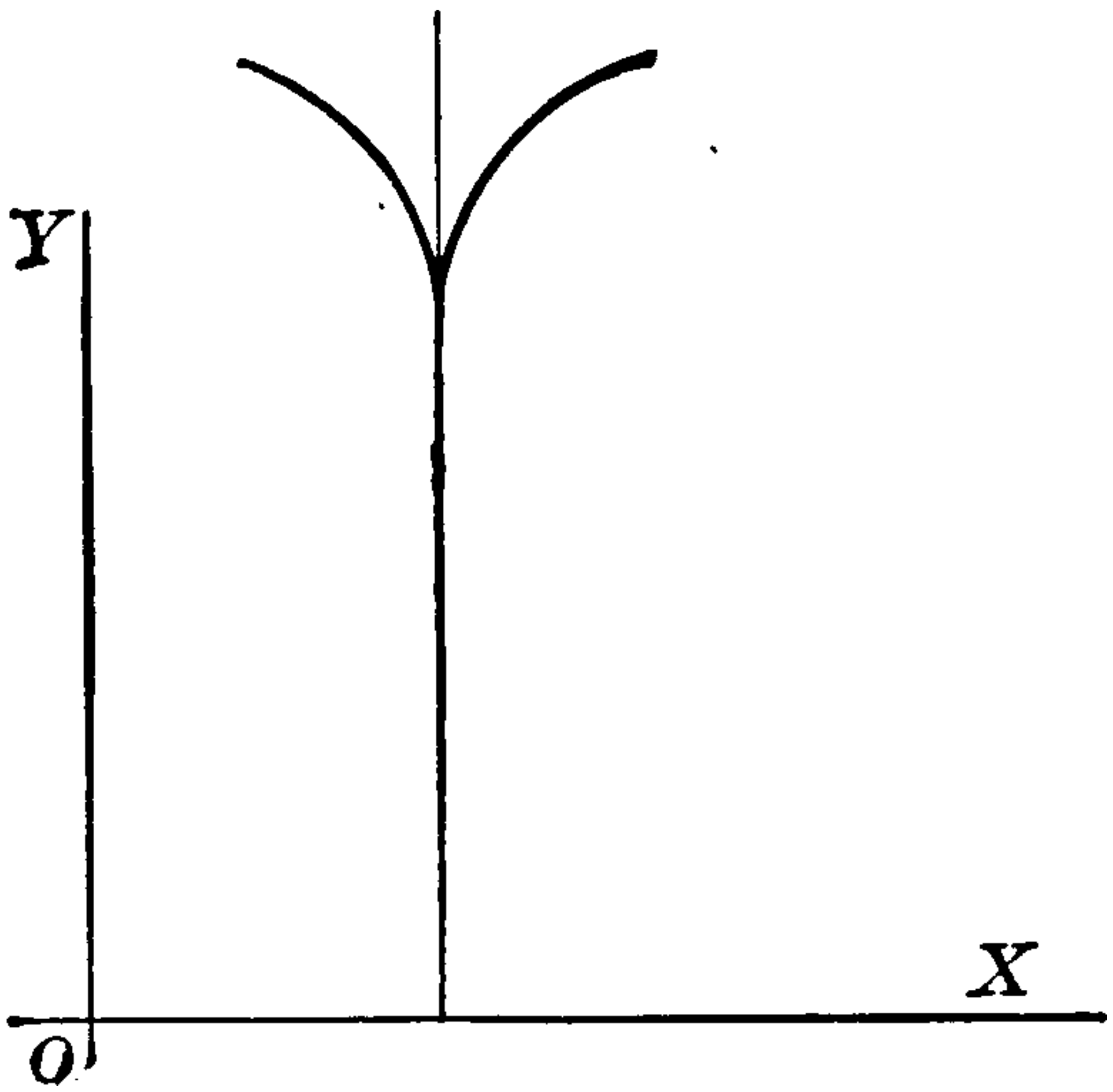


FIG. 16.

Ex. **3.** Examine for maxima and minima the function

$$\phi(x) = (x-1)^{\frac{1}{3}} + 1.$$

Here $\phi'(x) = \dfrac{1}{3}\dfrac{1}{(x-1)^{\frac{2}{3}}}$,

hence $\phi'(x)$ never changes sign, but is always positive. There is accordingly no turning value. The curve $y = \phi(x)$ has a vertical tangent at the point (1, 1), since $\dfrac{dy}{dx} = \phi'(x)$ is infinite when $x = 1$. (Fig. 17.)

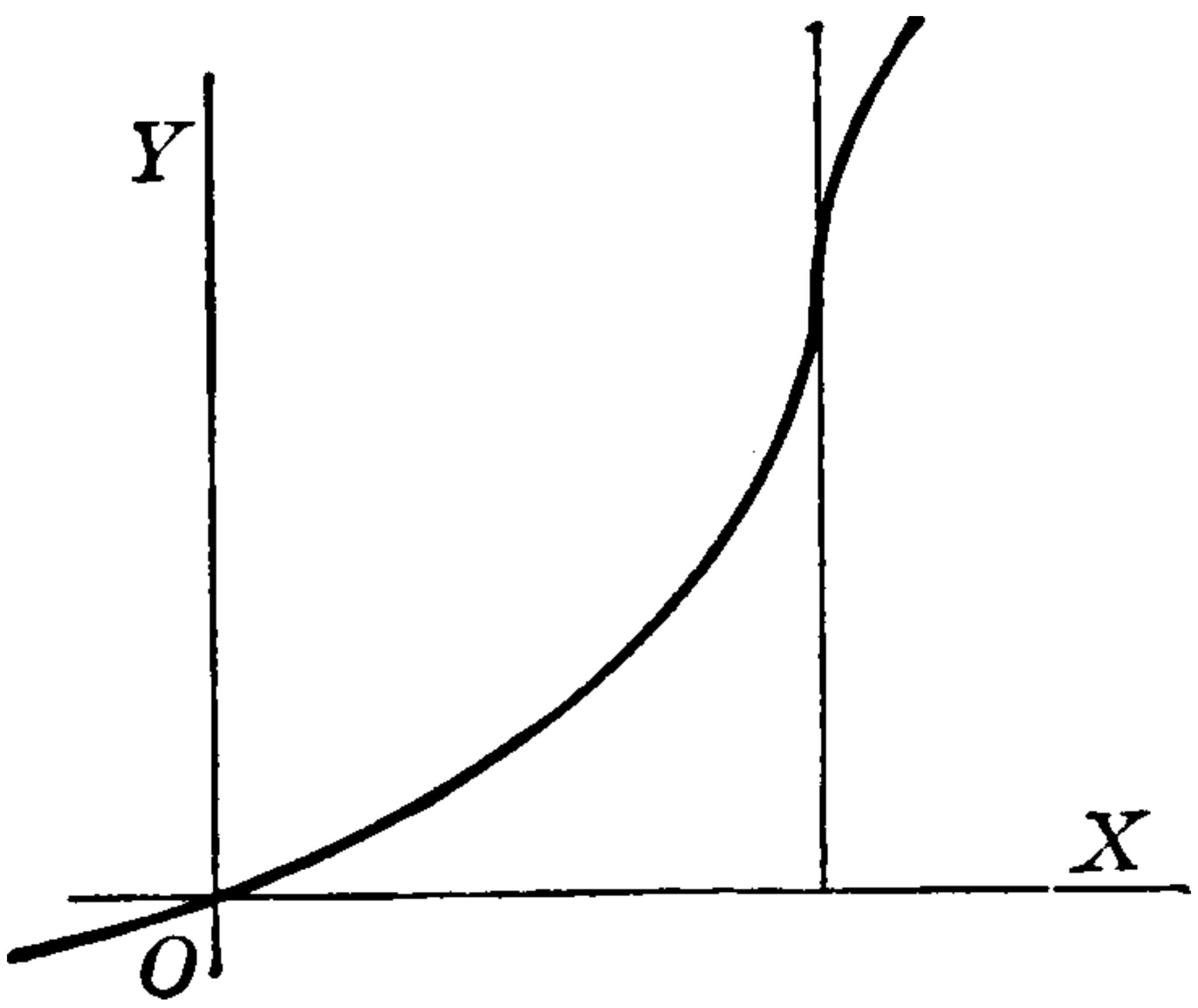


FIG. 17.

57. Critical values of the variable. It has been shown that the necessary and sufficient condition for a turning value of $\phi(x)$ is that $\phi'(x)$ shall change its sign. Now a function can change its sign only when it passes through zero, as in Ex. 1 (Art. 56), or when its reciprocal passes through zero, as in Exs. 2, 3. In the latter case it is usual to say that the function passes through infinity. It is not true, conversely, that a function always changes its sign in passing through zero or infinity, *e.g.*, $y = x^2$.

Nevertheless all the values of x, at which $\phi'(x)$ passes through zero or infinity, are called *critical* values of x, because they are to be further examined to determine whether $\phi'(x)$ actually changes sign as x passes through each such value; and whether, in consequence, $\phi(x)$ passes through a turning value.

For instance, in Ex. 1, the derivative $\phi'(x)$ vanishes when $x = -1$, and when $x = 2$, and it does not become infinite for any finite value of x. Thus the critical values are $-1, 2$, both of which give turning values to $\phi(x)$. Again, in Exs. 2, 3, the critical value is $x = 1$, since it makes $\phi'(x)$ infinite; it gives a turning value to $\phi(x)$ in Ex. 2, but not in Ex. 3.

58. Method of determining whether $\phi'(x)$ changes its sign in passing through zero or infinity. Let a be a critical value of x, in other words let $\phi'(a)$ be either zero or infinite, and let h be a very small positive number, so that $a-h$ and $a+h$ are two numbers very close to a, and on opposite sides of it. In order to determine whether $\phi'(x)$ changes sign as x increases through the value a, it is only necessary to compare the signs of $\phi'(a+h)$ and $\phi'(a-h)$. If it is possible to take h so small that $\phi'(a-h)$ is positive and $\phi'(a+h)$ negative, then $\phi'(x)$ changes sign as x passes through the

[illegible]

differentiate once, & let it = 0

value a, and $\phi(x)$ passes through a maximum value $\phi(a)$. Similarly, if $\phi'(a-h)$ is negative and $\phi'(a+h)$ positive, then $\phi(x)$ passes through a minimum value $\phi(a)$.

If $\phi'(a-h)$ and $\phi'(a+h)$ have the same sign, however small h may be, then $\phi(a)$ is not a turning value of $\phi(x)$.

Ex. Find the turning values of the function

$$\phi(x)=(x-1)^2(x+1)^3.$$

Here
$$\begin{aligned}\phi'(x)&=2(x-1)(x+1)^3+3(x-1)^2(x+1)^2\\&=(x-1)(x+1)^2(5x-1).\end{aligned}$$

Hence $\phi'(x)$ becomes zero at $x=-1, \frac{1}{5}$, and 1; it does not become infinite for any finite value of x.

Thus, the critical values are $-1, \frac{1}{5}, 1$.

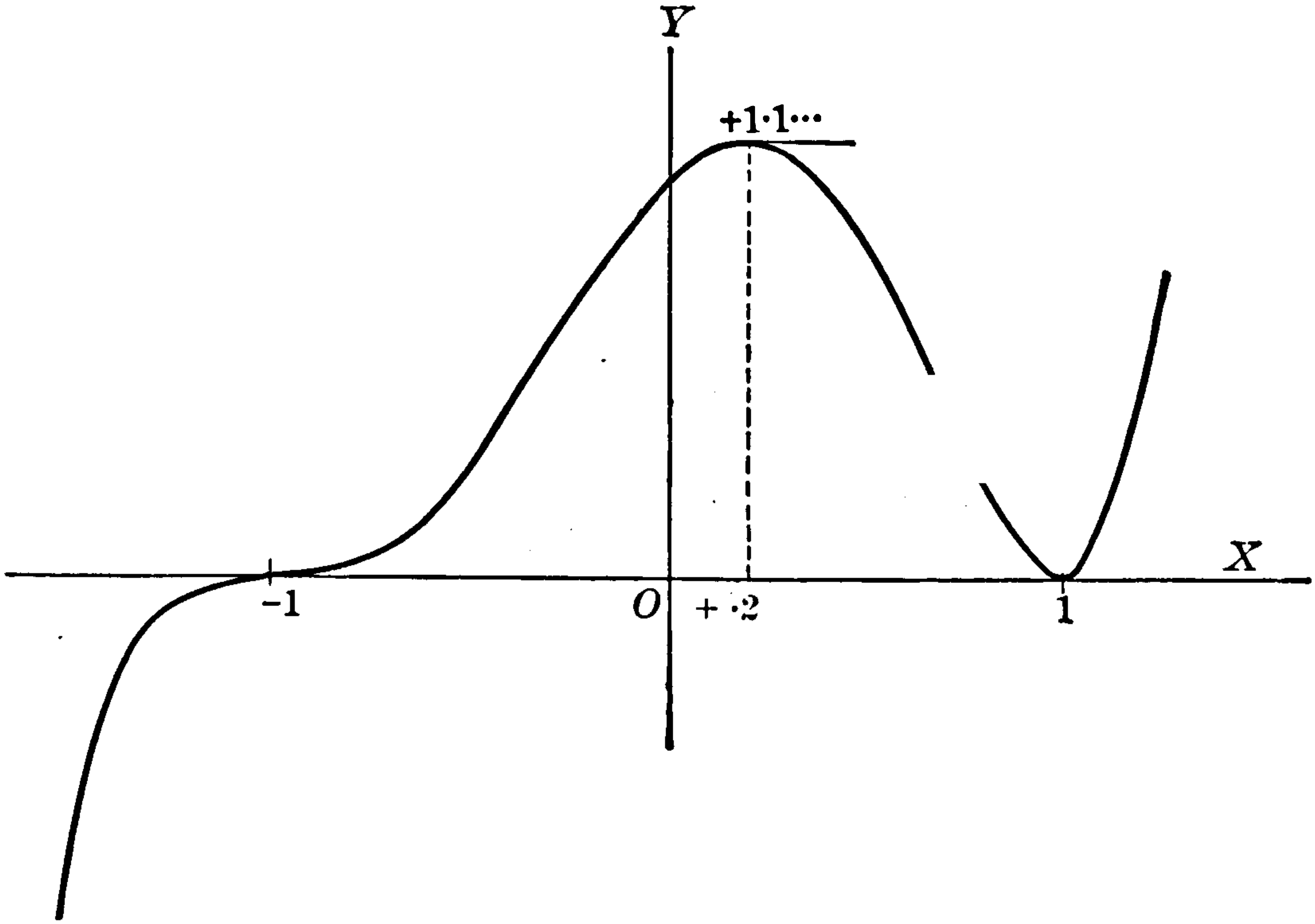


FIG. 18.

When $x=-1-h$, the three factors of $\phi'(x)$ take the signs $-\ +\ -$,
and when $x=-1+h$, they become $-\ +\ -$;
thus $\phi'(x)$ does not change sign as x increases through -1; hence $\phi(-1)=0$ is not a turning value of $\phi(x)$.

When $x=\frac{1}{5}-h$, the three factors of $\phi'(x)$ have signs $-\ +\ -$,
and when $x=\frac{1}{5}+h$, they become $-\ +\ +$;
thus $\phi'(x)$ changes sign from $+$ to $-$ as x increases through $\frac{1}{5}$, and $\phi(\frac{1}{5})=1\cdot11052$ is a maximum value of $\phi(x)$.

Finally, when $x = 1 - h$, the three factors of $\phi'(x)$ have the signs $- \; + \; +$, and when $x = 1 + h$ they become $+ \; + \; +$; thus $\phi'(x)$ changes sign from $-$ to $+$ as x increases through 1, and $\phi(1) = 0$ is a minimum value of $\phi(x)$.

The deportment of the function and its first derivative in the vicinity of the critical values may be tabulated as follows, in which inc., dec. stand for increasing, decreasing, respectively:

x	$-1-h$	-1	$-1+h$	$\frac{1}{5}-h$	$\frac{1}{5}$	$\frac{1}{5}+h$	$1-h$	1	$1+h$
$\phi'(x)$	$+$	0	$+$	$+$	0	$-$	$-$	0	$+$
$\phi(x)$	inc.		inc.	inc.	max.	dec.	dec.	min.	inc.

The general march of the function may be exhibited graphically by tracing the curve $y = \phi(x)$ (Fig. 18), using the foregoing result and observing the following simultaneous values of x and y:

$$x = -\infty, \; -2, \; -1, \; 0, \; \tfrac{1}{5}, \; 1, \; 2, \; \infty.$$

$$y = -\infty, \; -9, \; 0, \; 1, \; 1 \cdot 1 \cdots, \; 0, \; 27, \; \infty.$$

59 Second method of determining whether $\phi'(x)$ changes sign in passing through zero. The following method may be employed when the function and its derivatives are continuous in the vicinity of the critical value $x = a$.

Suppose, when x increases through the value a, that $\phi'(x)$ changes sign from positive through zero to negative. Its change from positive to zero is a decrease, and so is the change from zero to negative; thus $\phi'(x)$ is a decreasing function at $x = a$, and hence its derivative $\phi''(x)$ is negative at $x = a$.

On the other hand, if $\phi'(x)$ changes sign from negative through zero to positive, it is an increasing function and $\phi''(x)$ is positive at $x = a$; hence:

The function $\phi(x)$ has a maximum value $\phi(a)$, when $\phi'(a) = 0$ and $\phi''(a)$ is negative; $\phi(x)$ has a minimum value $\phi(a)$, when $\phi'(a) = 0$ and $\phi''(a)$ is positive.

It may happen, however, that $\phi''(a)$ is also zero.

In this case, to determine whether $\phi(x)$ has a turning value, it is necessary to proceed to the higher derivatives. If $\phi(x)$ is a maximum, $\phi''(x)$ is negative just before vanishing, and negative just after, for the reason given above; but the change from negative to zero is an increase, and the change from zero to negative is a decrease; thus $\phi''(x)$ changes from increasing to decreasing as x passes through a. Hence $\phi'''(x)$ changes sign from positive through zero to negative, and it follows, as before, that its derivative $\phi^{\text{IV}}(x)$ is negative.

Thus $\phi(a)$ is a maximum value of $\phi(x)$ if $\phi'(a) = 0$, $\phi''(a) = 0$, $\phi'''(a) = 0$, $\phi^{\text{IV}}(a)$ negative. Similarly, $\phi(a)$ is a minimum value of $\phi(x)$ if $\phi'(a) = 0$, $\phi''(a) = 0$, $\phi'''(a) = 0$, and $\phi^{\text{IV}}(a)$ positive.

If it happen that $\phi^{\text{IV}}(a) = 0$, it is necessary to proceed to still higher derivatives to test for turning values. The result may then be generalized as follows:

The function $\phi(x)$ has a maximum (or minimum) value at $x = a$ if one or more of the derivatives $\phi'(a)$, $\phi''(a)$, $\phi'''(a)$ vanish and if the first one that does not vanish is of even order, and negative (or positive).

Ex. Find the critical values in the example of Art. 58 by the second method.

$$\phi''(x) = (x+1)^2(5x-1) + 2(x-1)(x+1)(5x-1) + 5(x-1)(x+1)^2,$$
$$= 4(5x^3 + 3x^2 - 3x - 1),$$

$\phi''(1) = 16$, hence $\phi(1)$ is a minimum value of $\phi(x)$,

$\phi''(-1) = 0$, hence it is necessary to find $\phi'''(-1)$;

$\phi'''(x) = 12(5x^2 + 2x - 1)$,

$\phi'''(-1) = 24$, hence $\phi(-1)$ is neither a maximum nor a minimum value of $\phi(x)$.

Again, $\phi''(\frac{1}{5}) = 5(\frac{1}{5} - 1)(\frac{1}{5} + 1)^2$ is negative, hence $\phi(\frac{1}{5})$ is a maximum value of $\phi(x)$.

60. Conditions for maxima and minima derived from Taylor's theorem. In this article, as in the preceding, the function and its derivatives are supposed to be continuous in the vicinity of $x = a$; otherwise the method of Art. 58 must be used.

If $\phi(a)$ be a maximum value of $\phi(x)$, it follows from the definition that $\phi(a)$ is greater than either of the neighboring values, $\phi(a + h)$, or $\phi(a - h)$, when h is taken small enough. Hence $\phi(a + h) - \phi(a)$ and $\phi(a - h) - \phi(a)$ are both negative.

Similarly, these expressions are both positive if $\phi(a)$ is a minimum value of $\phi(x)$.

Let $\phi(x + h)$ and $\phi(x - h)$ be expanded in powers of h by Taylor's theorem.

Then $$\phi(x+h) = \phi(x) + \phi'(x)h + \frac{\phi''(x)}{2\,!}h^2 + \frac{\phi'''(x)}{3\,!}h^3 + \cdots,$$

$$\phi(x-h) = \phi(x) - \phi'(x)h + \frac{\phi''(x)}{2\,!}h^2 - \frac{\phi'''(x)}{3\,!}h^3 + \cdots.$$

If x be replaced by a, and $\phi(a)$ transposed, the result is

$$\phi(a+h) - \phi(a) = \phi'(a)h + \frac{\phi''(a)}{2\,!}h^2 + \frac{\phi'''(a)}{3\,!}h^3 + \cdots,$$

$$\phi(a-h) - \phi(a) = -\phi'(a)h + \frac{\phi''(a)}{2\,!}h^2 - \frac{\phi'''(a)}{3\,!}h^3 + \cdots.$$

The increment h can now be taken so small that $h\phi'(a)$ will be numerically larger than the sum of the remaining terms in the second member of either of the last two equations, and its sign will therefore determine the sign of the entire member. Since these signs are opposite in the two equations, $\phi(a + h) - \phi(a)$ and $\phi(a - h) - \phi(a)$ cannot have the same sign unless $\phi'(a)$ is zero, hence the first condition for a turning value is $\phi'(a) = 0$.

In case $\phi'(a) = 0$ the preceding equations become

$$\phi(a+h) - \phi(a) = \frac{\phi''(a)}{2!}h^2 + \frac{\phi'''(a)}{3!}h^3 + \cdots,$$

$$\phi(a-h) - \phi(a) = \frac{\phi''(a)}{2!}h^2 - \frac{\phi'''(a)}{3!}h^3 + \cdots,$$

and h can be taken so small that the first term on the right is numerically larger than either of the second terms, hence $\phi(a+h) - \phi(a)$ and $\phi(a-h) - \phi(a)$ are both negative when $\phi''(a)$ is negative, and both positive when $\phi''(a)$ is positive.

Thus $\phi(a)$ is a maximum (or minimum) value of $\phi(x)$ when $\phi'(a)$ is zero and $\phi''(a)$ is negative (or positive).

If it should happen that $\phi''(a)$ is also zero, then

$$\phi(a+h) - \phi(a) = \frac{\phi'''(a)}{3!}h^3 + \frac{\phi^{\text{IV}}(a)}{4!}h^4 + \cdots,$$

$$\phi(a-h) - \phi(a) = -\frac{\phi'''(a)}{3!}h^3 + \frac{\phi^{\text{IV}}(a)}{4!}h^4 + \cdots,$$

and by the same reasoning as before, it follows that for a maximum (or minimum) there are the further conditions that $\phi'''(a)$ equals zero, and that $\phi^{\text{IV}}(a)$ is negative (or positive).

Proceeding in this way, the general conclusion stated in the last article is evident.

Ex. 1. Which of the preceding examples can be solved by the general rule here referred to?

Ex. 2. Why was the restriction imposed upon $\phi'(x)$ that it should change sign by passing through zero, rather than by passing through infinity?

61. The maxima and minima of any continuous function occur alternately. It has been seen that the maximum and minimum values of a rational polynomial occur alternately when the variable is continually increased, or diminished.

This principle is also true in the case of every continuous function of a single variable. For, let $\phi(a)$, $\phi(b)$ be two maximum values of $\phi(x)$, in which a is supposed less than b. Then, when $x = a + h$, the function is decreasing; when $x = b - h$, the function is increasing, h being taken sufficiently small and positive. But in passing from a decreasing to an increasing state, a continuous function must, at some intermediate value of x, change from decreasing to increasing, that is, must pass through a minimum. Hence, between two maxima there must be at least one minimum.

It can be similarly proved that between two minima there must be at least one maximum.

62. Simplifications that do not alter critical values. The work of finding the critical values of the variable, in the case of any given function, may often be simplified by means of the following self-evident principles.

1. When c is independent of x, any value of x that gives a turning value to $c\phi(x)$ gives also a turning value to $\phi(x)$; and conversely. These two turning values are of the same or opposite kind according as c is positive or negative.

2. Any value of x that gives a turning value to $c + \phi(x)$ gives also a turning value of the same kind to $\phi(x)$; and conversely.

3. When n is independent of x, any value of x that gives a turning value to $[\phi(x)]^n$ gives also a turning value to $\phi(x)$; and conversely. Whether these turning values are of the same or opposite kind depends on the sign of n, and also on the sign of $[\phi(x)]^{n-1}$.

EXERCISES

Find the critical values of x in the following functions, determine the nature of the function at each, and obtain the graph of the function.

1. $u = x(x^2 - 1)$.

2. $u = 2x^3 - 15x^2 + 36x - 4$.

3. $u = (x-1)^3(x-2)^2$.

4. $u = \sin x + \cos x$.

5. $u = \dfrac{(a-x)^3}{a-2x}$.

6. $u = x(x+1)^2 - 5$.

7. $u = 5 + 12x - 3x^2 - 2x^3$.

8. $u = \dfrac{\log x}{x}$.

9. $u = \sin^2 x \cos^3 x$.

10. Show that a quadratic integral function always has one maximum, or one minimum, but never both.

11. Show that a cubic integral function has in general both a maximum and a minimum value, but may have neither.

12. Show that the function $(x - b)^{\frac{5}{3}}$ has neither a maximum nor a minimum value.

63. Geometric problems in maxima and minima. The theory of the turning values of a function has important applications in solving problems concerning geometric maxima or minima, *i.e.*, the determination of the largest or the smallest value a magnitude may have while satisfying certain stated geometric conditions.

The first step is to express the magnitude in question algebraically. If the resulting expression contains more than one variable, the stated conditions will furnish enough relations between these variables, so that all the others may be expressed in terms of one. The expression to be maximized or minimized, being thus made a function of a single variable, can be treated by the preceding rules.

Ex. 1. Find the largest rectangle whose perimeter is 100. Let x, y denote the dimensions of any of the rectangles whose perimeter is 100. The expression to be maximized is the area

$$u = xy, \tag{1}$$

in which the variables x, y are subject to the stated condition

$$2x + 2y = 100,$$

i.e.,

$$y = 50 - x; \tag{2}$$

hence the function to be maximized, expressed in terms of the single variable x, is

$$u = \phi(x) = x(50 - x) = 50x - x^2. \tag{3}$$

The critical value of x is found from the equation

$$\phi'(x) = 50 - 2x = 0$$

to be $x = 25$. When x increases through this value, $\phi'(x)$ changes sign from positive to negative, and hence $\phi(x)$ is a maximum when $x = 25$. Equation (2) shows that the corresponding value of y is 25. Hence the maximum rectangle whose perimeter is 100 is the square whose side is 25.

Ex. 2. If, from a square piece of tin whose side is a, a square be cut out at each corner, find the side of the latter square in order that the remainder may form a box of maximum capacity, with open top.

Let x be a side of each square cut out. Then the bottom of the box will be a square whose side is $a - 2x$, and the depth of the box will be x. Hence the volume is

$$v = x(a - 2x)^2,$$

which is to be made a maximum by varying x.

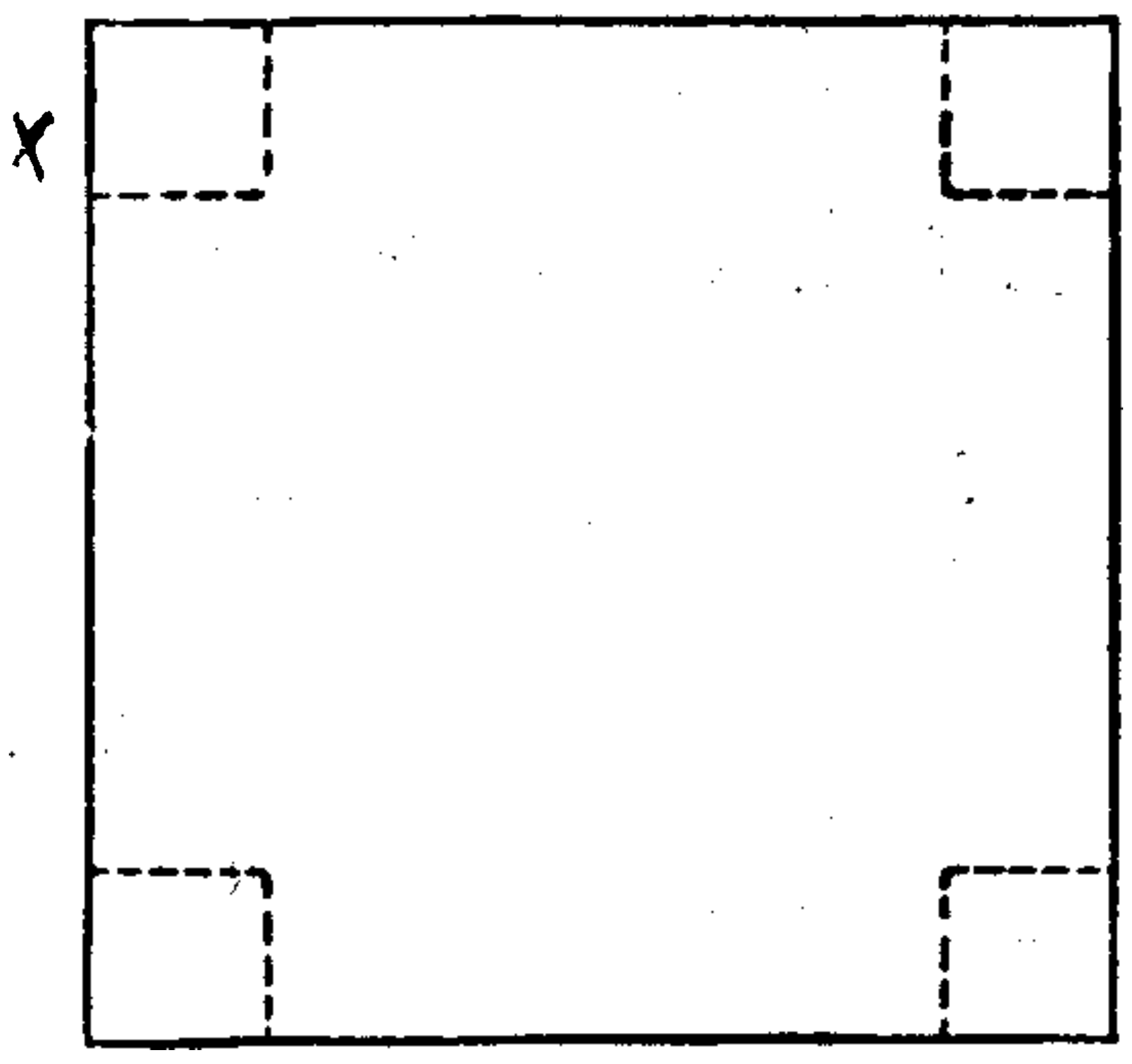


FIG. 19.

Here
$$\frac{dv}{dx} = (a - 2x)^2 - 4x(a - 2x)$$
$$= (a - 2x)(a - 6x).$$

This derivative vanishes when $x = \frac{a}{2}$, and when $x = \frac{a}{6}$. It will be found, by applying the usual test, that $x = \frac{a}{2}$ gives v the minimum value zero, and that $x = \frac{a}{6}$ gives it the maximum value $\frac{2a^3}{27}$. Hence the side of the square to be cut out is one sixth the side of the given square.

Ex. **3.** Find the area of the greatest rectangle that can be inscribed in a given ellipse.

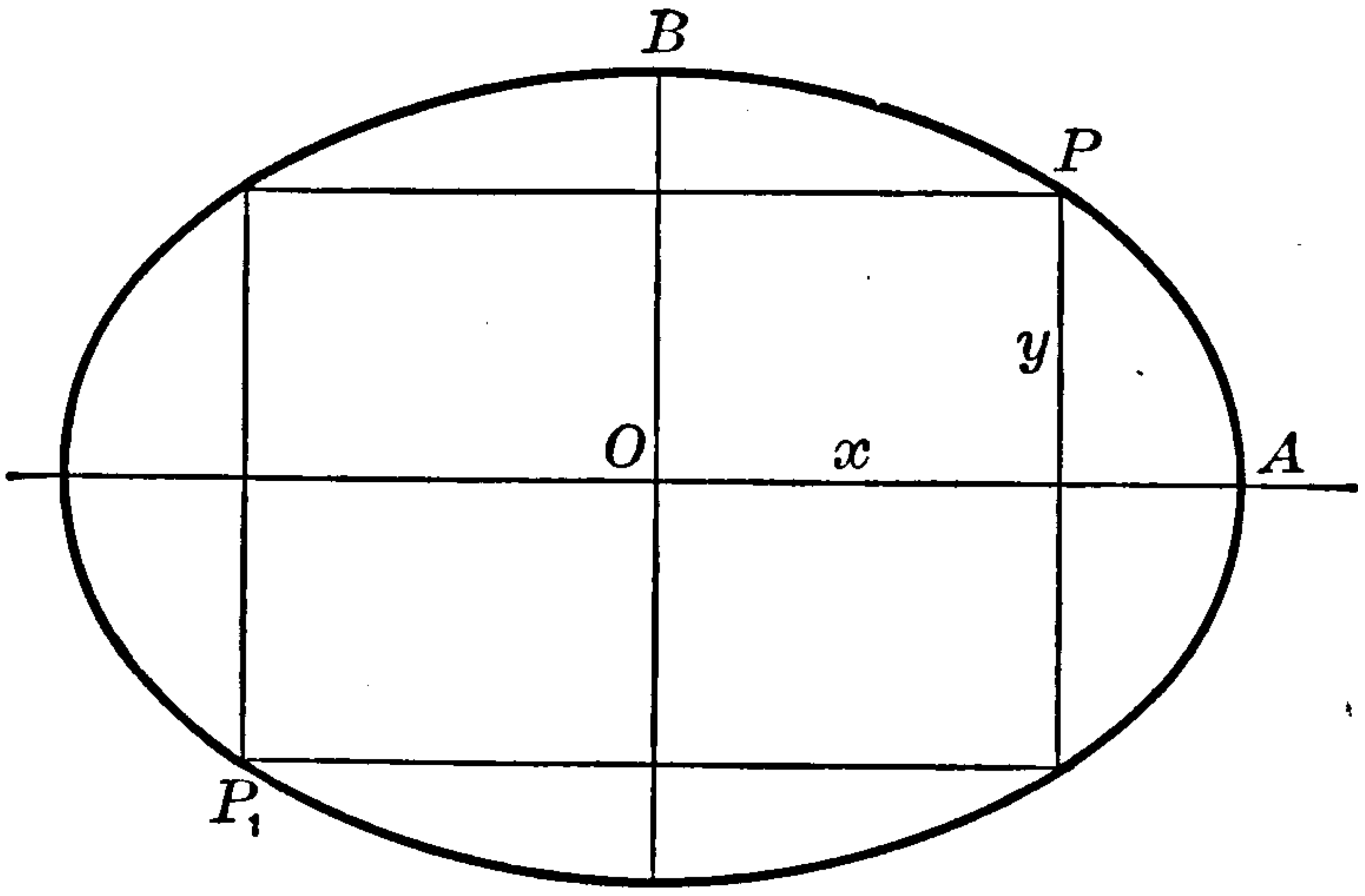


FIG. 20.

An inscribed rectangle will evidently be symmetric with regard to the principal axes of the ellipse.

Let a, b denote the lengths of the semi-axes OA, OB (Fig. 20); let $2x$, $2y$ be the dimensions of an inscribed rectangle. Then the area is

$$u = 4xy, \tag{1}$$

in which the variables x, y may be regarded as the coördinates of the vertex P, and are therefore subject to the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{2}$$

It is geometrically evident that there is some position of P for which the inscribed rectangle is a maximum.

The elimination of y from (1), by means of (2), gives the function of x to be maximized,

$$u=\frac{4b}{a}x\sqrt{a^2-x^2}. \tag{3}$$

By Art. 62, the critical values of x are not altered if this function be divided by the constant $\frac{4b}{a}$, and then squared. Hence, the values of x which render u a maximum, give also a maximum value to the function

$$\phi(x) = x^2(a^2-x^2) = a^2x^2-x^4.$$

Here $\phi'(x) = 2a^2x-4x^3 = 2x(a^2-2x^2),$

$$\phi''(x) = 2a^2-12x^2;$$

hence, by the usual tests, the critical values $x=\pm\frac{a}{\sqrt{2}}$ render $\phi(x)$, and therefore the area u, a maximum. The corresponding values of y are given by (2), and the vertex P may be at any of the four points denoted by

$$x=\pm\frac{a}{\sqrt{2}},\quad y=\pm\frac{b}{\sqrt{2}},$$

giving in each case the same maximum inscribed rectangle, whose dimensions are $a\sqrt{2}$, $b\sqrt{2}$, and whose area is $2ab$, or half that of the circumscribed rectangle.

Ex. 4. Find the greatest cylinder that can be cut from a given right cone, whose height is h, and the radius of whose base is a.

Let the cone be generated by the revolution of the triangle OAB (Fig. 21), and the inscribed cylinder be generated by the revolution of the rectangle AP.

Let $OA = h$, $AB = a$, and let the coördinates of P be (x, y). Then the function to be maximized is $\pi y^2(h-x)$ subject to the relation $\frac{y}{x} = \frac{a}{h}$.

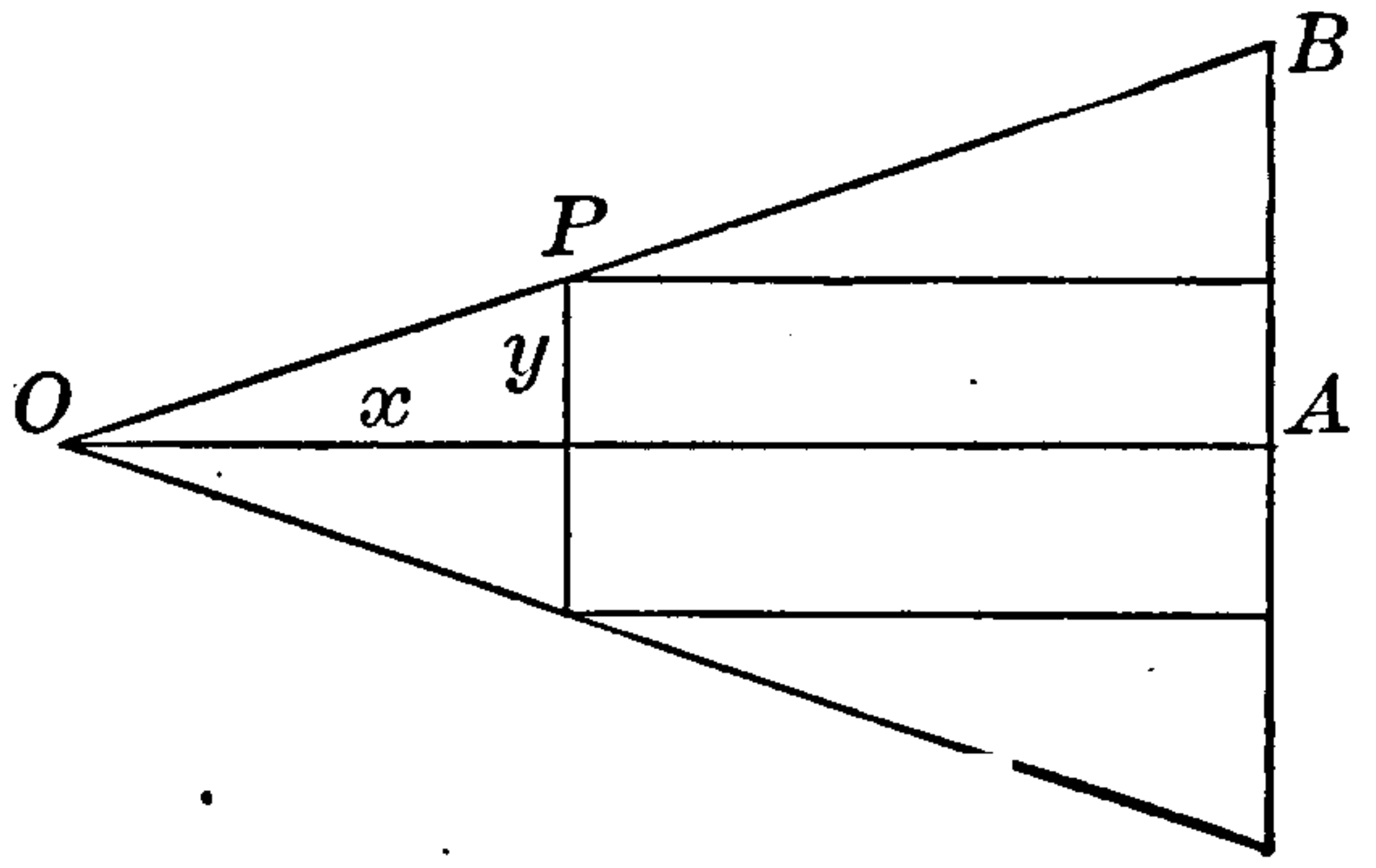


FIG. 21.

This expression becomes

$$V = \frac{\pi a^2}{h^2} \cdot x^2(h - x).$$

The critical value of x is $\frac{2}{3}h$, and $V = \frac{4\pi a^2 h}{27}$.

EXERCISES ON CHAPTER VI

1. Through a given point within an angle draw a straight line which shall cut off a minimum triangle. Solve this problem by the method of the calculus, and also by geometry.

[Take given lines as coördinate axes.]

2. The volume of a cylinder being constant, find its form when the entire surface is a minimum.

3. A rectangular court is to be built so as to contain a given area c^2, and a wall already constructed is available for one of its sides. Find its dimensions so that the expense incurred may be the least possible.

4. The sum of the surfaces of a sphere and a cube is given. How do their volumes compare when the sum of their volumes is a minimum?

5. What is the length of the axis of the maximum parabola which can be cut from a given right circular cone, given that the area of a parabola is equal to two thirds of the product of its base and altitude?

6. Determine the greatest rectangle which can be inscribed in a given triangle whose base is $2b$ and whose altitude is a.

7. The flame of a candle is directly over the center of a circle whose radius is 5 inches. What ought to be the height of the flame above the plane of the circle so as to illuminate the circumference as much as possible, supposing the intensity of the light to vary directly as the sine of the angle under which it strikes the illuminated surface, and inversely as the square of its distance from the illuminated point?

✓ **8.** A rectangular piece of pasteboard 30 inches long and 14 inches wide has a square cut out at each corner. Find the side of this square so that the remainder may form a box of maximum contents.

9. Find the altitude of the right cylinder of greatest volume inscribed in a sphere whose radius is r.

10. Through the point (a, b) a line is drawn such that the part intercepted between the rectangular coördinate axes is a minimum. Find its length.

✓ **11.** Given the slant height a of a right cone; find its altitude when the volume is a maximum.

12. The radius of a circular piece of paper is r. Find the arc of the sector which must be cut from it so that the remaining sector may form the convex surface of a cone of maximum volume.

13. Find relation between length of circular arc and radius in order that the area of a circular sector of given perimeter should be a maximum.

14. On the line joining the centers of two spheres of radii r, R, find the distance of the point from the center of the first sphere from which the maximum of spherical surface is visible.

15. Describe a circle with its center on a given circle so that the length of the arc intercepted within the given circle shall be a maximum.

CHAPTER VII

RATES AND DIFFERENTIALS

64. Rates. Time as independent variable. Suppose a particle P is moving in any path, straight or curved, and let s be the number of space units passed over in t seconds. Then s may be taken as the dependent variable, and t as the independent variable.

If Δs be the number of space units described in the additional time Δt seconds, then the *average velocity* of P during the time Δt is $\frac{\Delta s}{\Delta t}$; that is, the average number of space units described per second during the interval.

The velocity of P is said to be *uniform* if its average velocity $\frac{\Delta s}{\Delta t}$ is the same for all intervals Δt. The *actual velocity* of P at any instant of time t is the limit which the average velocity approaches as Δt is made to approach zero as a limit.

Thus
$$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

is the actual velocity of P at the time denoted by t. It is evidently the number of space units that would be passed over in the next second if the velocity remained uniform from the time t to the time $t+1$.

It may be observed that if the more general term, "rate of change," be substituted for the word "velocity," the above statements will apply to any quantity that varies with the time, whether it be length, volume, strength of current, or any other function of the time. For instance, let the quantity of an electric current be C at the time t, and $C+\Delta C$ at the time $t+\Delta t$. Then the *average rate of change* of current in the interval Δt is $\frac{\Delta C}{\Delta t}$; this is the average increase in current-units per second. And the *actual rate of change* at the instant denoted by t is

$$\lim_{\Delta t \doteq 0} \frac{\Delta C}{\Delta t} = \frac{dC}{dt}.$$

This is the number of current-units that would be gained in the next second if the rate of gain were uniform from the time t to the time $t+1$. Since, by Art. 14,

$$\frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt},$$

hence $\frac{dy}{dx}$ measures the ratio of the rates of change of y and of x.

It follows that the result of differentiating

$$y = f(x) \tag{1}$$

may be written in either of the forms

$$\frac{dy}{dx} = f'(x), \tag{2}$$

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}. \tag{3}$$

The latter form is often convenient, and may also be obtained directly from (1) by differentiating both sides with regard to t. It may be read: the rate of change of y is $f'(x)$ times the rate of change of x.

Returning to the illustration of a moving point P, let its coördinates at time t be x and y. Then $\frac{dx}{dt}$ measures the rate of change of the x-coördinate.

Since velocity has been defined as the rate at which a point is moving, the rate $\frac{dx}{dt}$ may be called the velocity which the point P has in the direction of the x-axis, or, more briefly, the x-component of the velocity of P.

It was shown on p. 105 that the actual velocity at any instant t is equal to the space that would be passed over in a unit of time, provided the velocity were uniform during that unit. Accordingly, the x-component of velocity $\frac{dx}{dt}$ may be represented by the distance PA (Fig. 22) which P would pass over in the direction of the x-axis during a unit of time if the velocity remained uniform.

Similarly $\frac{dy}{dt}$ is the y-component of the velocity of P, and may be represented by the distance PB.

The velocity $\frac{ds}{dt}$ of P along the curve can be represented by the distance PC, measured on the tangent line to the curve at P. It is evident that PC is the diagonal of the rectangle PA, PB.

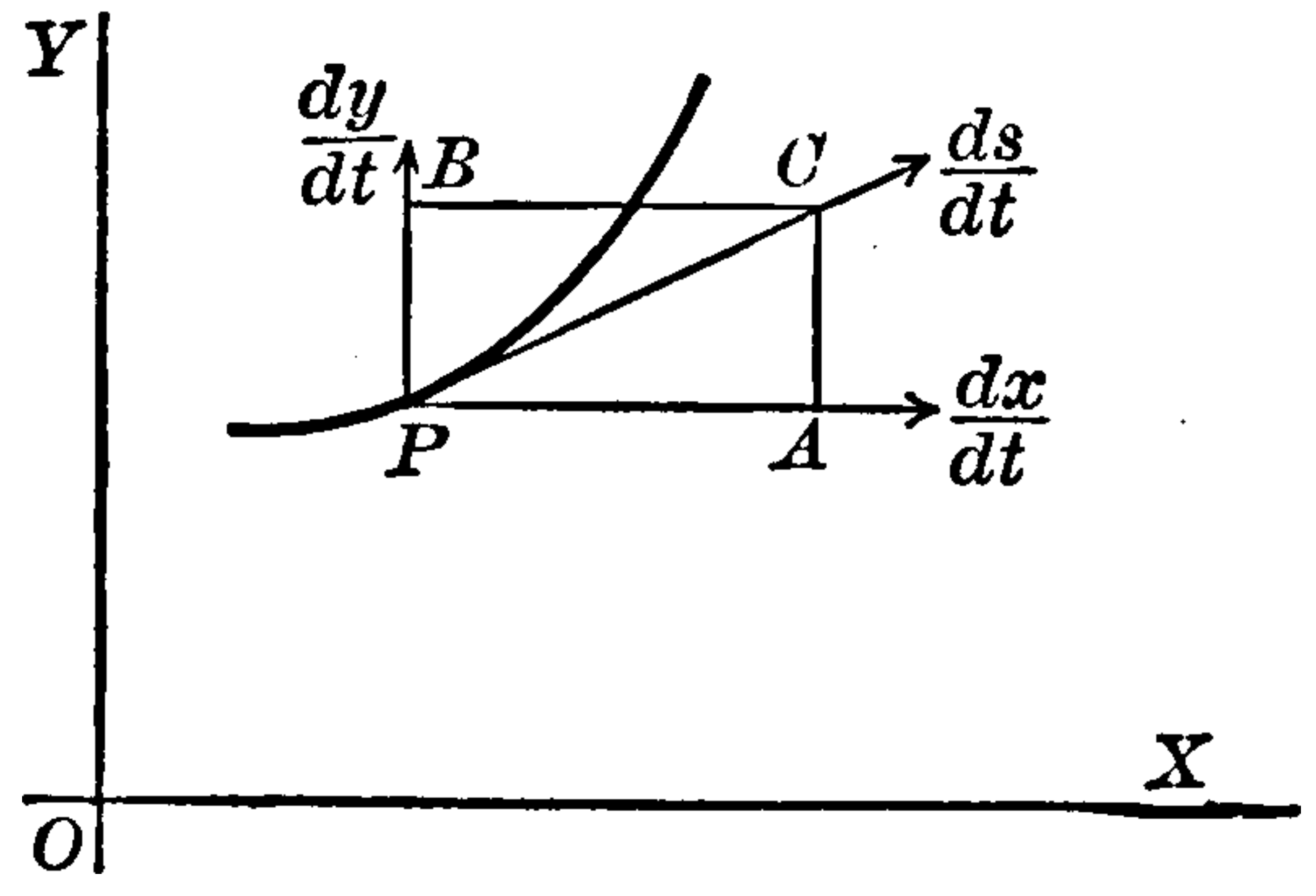


FIG. 22.

Since $PC^2 = PA^2 + PB^2$,

it follows that

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \qquad (4)$$

Ex. 1. If a point describe the straight line $3x + 4y = 5$, and if x increase h units per second, find the rates of increase of y and of s.

Since $$y = \tfrac{5}{4} - \tfrac{3}{4}x,$$

hence $$\frac{dy}{dt} = -\frac{3}{4}\frac{dx}{dt}.$$

When $$\frac{dx}{dt} = h,$$

it follows that $\frac{dy}{dt} = -\tfrac{3}{4}h$, $\quad \frac{ds}{dt} = \sqrt{h^2 + \tfrac{9}{16}h^2} = \tfrac{5}{4}h.$

Ex. **2**. A point describes the parabola $y^2 = 12x$ in such a way that when $x = 3$ the abscissa is increasing at the rate of 2 feet per second; at what rate is y then increasing? Find also the rate of increase of s.

Since $$y^2 = 12x,$$

then $$2y\frac{dy}{dt} = 12\frac{dx}{dt},$$

$$\frac{dy}{dt} = \frac{6}{y}\frac{dx}{dt} = \frac{6}{\sqrt{12x}}\frac{dx}{dt};$$

hence, when $x = 3$, and $\frac{dx}{dt} = 2$, it follows that $\frac{dy}{dt} = \pm 2$.

Again, $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, hence $\frac{ds}{dt} = 2\sqrt{2}$ feet per second.

Ex. **3**. A person is walking toward the foot of a tower on a horizontal plane at the rate of 5 miles per hour; at what rate is he approaching the top, which is 60 feet high, when he is 80 feet from the bottom?

Let x be the distance from the foot of the tower at time t, and y the distance from the top at the same time. Then

$$x^2 + 60^2 = y^2,$$

and $$x\frac{dx}{dt} = y\frac{dy}{dt}.$$

When x is 80 feet, y is 100 feet; hence if $\frac{dx}{dt}$ is 5 miles per hour, $\frac{dy}{dt}$ is 4 miles per hour.

65. Abbreviated notation for rates. When, as in the above examples, a time derivative is a factor of each member of an equation, it is usually convenient to write, instead of the symbols $\frac{dx}{dt}, \frac{dy}{dt}$, the abbreviations dx and dy, for the rates of change of the variables x and y. Thus the result of differentiating

$$y = f(x) \tag{1}$$

may be written in either of the forms

$$\frac{dy}{dx} = f'(x), \tag{2}$$

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}, \tag{3}$$

$$dy = f'(x)\,dx. \tag{4}$$

It is to be observed that the last form is not to be regarded as derived from equation (2) by separation of the symbols dy, dx; for the derivative $\frac{dy}{dx}$ has been defined as the result of performing upon y an indicated operation represented by the symbol $\frac{d}{dx}$; and thus the dy and dx of the symbol $\frac{dy}{dx}$ have been given no separate meaning. The dy and dx of equation (4) stand for the rates, or time derivatives, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ occurring in (3), while the latter equation is itself obtained from (1) by differentiation with regard to t, by Art. 14.

In case the dependence of y upon x be not indicated by a functional operation f, equations (3), (4) take the form

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt},$$

$$dy=\frac{dy}{dx}dx.$$

In the abbreviated notation, equation (4) of the last article is written

$$ds^2=dx^2+dy^2.$$

Ex. 1. A point describing the parabola $y^2=2px$ is moving at the time t with a velocity of v feet per second. Find the rate of increase of the coördinates x and y at the same instant.

Differentiating the given equation with regard to t,

$$ydy=pdx.$$

But dx, dy also satisfy the relation

$$dx^2+dy^2=v^2;$$

hence, by solving these simultaneous equations,

$$dx=\frac{y}{\sqrt{y^2+p^2}}v,\quad dy=\frac{p}{\sqrt{y^2+p^2}}v, \text{ in feet per second.}$$

Ex. 2. A vertical wheel of radius 10 ft. is making 5 revolutions per second about a fixed axis. Find the horizontal and vertical velocities of a point on the circumference situated 30° from the horizontal.

Since $x = 10\cos\theta,\quad y = 10\sin\theta,$

then $dx = -10\sin\theta d\theta,\quad dy = 10\cos\theta d\theta.$

But $d\theta = 10\pi = 31.416$ radians per second,

hence $dx = -314.16\sin\theta = -157.08$ feet per second,

and $dy = 314.16\cos\theta = 272.06$ feet per second.

Ex. 3. Trace the changes in the horizontal and vertical velocity in a complete revolution.

66. Differentials often substituted for rates. The symbols dx, dy have been defined above as the rates of change of x and y per second.

Sometimes, however, they may conveniently be allowed to stand for any two numbers, large or small, that are proportional to these rates; the equations, being homogeneous in them, will not be affected. It is usual in such cases to speak of the numbers dx and dy by the more general name of *differentials;* they may then be either the rates themselves, or any two numbers in the same ratio.

This will be especially convenient in problems in which the time variable is not explicitly mentioned.

Ex. 1. When x increases from 45° to 45° 15′, find the increase of $\log_{10}\sin x$, assuming that the ratio of the rates of change of the function and the variable remains sensibly constant throughout the short interval.

Here $dy = \log_{10}e \cdot \cot x dx = .4343 \cot x dx = .4343\, dx.$

Let $dx = 15' = .004363$ radians.

Then $dy = .001895,$

which is the approximate increment of $\log_{10}\sin x$.

But $\log_{10}\sin 45° = -\frac{1}{2}\log 2 = -.150515,$

therefore $\log_{10}\sin 45°\,15' = -.148620.$

Ex. **2**. Expanding $\log_{10}\sin(x+h)$ as far as h^2 by Taylor's theorem, and then putting $x = .785398$, $h = .004363$, show what is the error made by neglecting the third term, as was done in Ex. 1.

Ex. **3**. When x varies from 60° to 60° 10′, find the increase in $\sin x$.

Ex. **4**. Show that $\log_{10}x$ increases more slowly than x, when $x > \log_{10}e$, that is, $x > .4343$.

Ex. **5**. Two sides a, b of a triangle are measured, and also the included angle C; find the error in the computed length of the third side c due to a small error in the observed angle C.

[Differentiate the equation $c^2 = a^2 + b^2 - 2\,ab\cos C$, regarding a, b as constant.]

✓ Ex. **6**. A vessel is sailing northwest at the rate of 10 miles per hour. At what rate is she making north latitude?

✓ Ex. **7**. In the parabola $y^2 = 12\,x$, find the point at which the ordinate and abscissa are increasing equally.

✓ Ex. **8**. At what part of the first quadrant does the angle increase twice as fast as its sine?

✓ Ex. **9**. Find the rate of change in the area of a square when the side b is increasing at a ft. per second.

✓ Ex. **10**. In the function $y = 2\,x^3 + 6$, what is the value of x at the point where y increases 24 times as fast as x?

✓ Ex. **11**. A circular plate of metal expands by heat so that its diameter increases uniformly at the rate of 2 inches per second; at what rate is the surface increasing when the diameter is 5 inches?

✓ Ex. **12**. What is the value of x at the point at which $x^3 - 5\,x^2 + 17\,x$ and $x^3 - 3\,x$ change at the same rate?

✓ Ex. **13**. Find the points at which the rate of change of the ordinate $y = x^3 - 6\,x^2 + 3\,x + 5$ is equal to the rate of change of the slope of the tangent to the curve.

✓ Ex. **14**. The relation between s, the space through which a body falls, and t, the time of falling, is $s = 16\,t^2$; show that the velocity is equal to $32\,t$.

The rate of change of velocity is called *acceleration;* show that the acceleration of the falling body is a constant.

✓ Ex. **15**. A body moves according to the law $s = \cos(nt + e)$. Show that its acceleration is proportional to the space through which it has moved.

✓ Ex. **16**. If a body be projected upwards in a vacuum with an initial velocity v_0, to what height will it rise, and what will be the time of ascent?

✓ Ex. **17**. A body is projected upwards with a velocity of a feet per second. After what time will it return?

✓ Ex. **18**. If A be the area of a circle of radius x, show that the circumference is $\frac{dA}{dx}$. Interpret this fact geometrically.

✓ Ex. **19**. A point describing the circle $x^2 + y^2 = 25$ passes through (3, 4) with a velocity of 20 feet per second. Find its component velocities parallel to the axes.

CHAPTER VIII

DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES

Thus far only functions of a single variable have been considered. The present chapter will be devoted to the study of functions of two independent variables x, y. They will be represented by the symbol

$$z = f(x, y).$$

If the simultaneous values of the three variables x, y, z be represented as the rectangular coördinates of a point in space, the locus of all such points is a surface having the equation $z = f(x, y)$.

67. Definition of continuity. A function z of x and y, $z = f(x, y)$, is said to be continuous in the vicinity of any point (a, b) when $f(a, b)$ is real, finite, and determinate, and such that

$$\lim_{\substack{h \doteq 0 \\ k \doteq 0}} f(a + h, b + k) = f(a, b),$$

however h and k approach zero.

When a pair of values a, b exists at which any one of these properties does not hold, the function is said to be discontinuous at the point (a, b).

E.g., let $$z = \frac{x + y}{x - y}.$$

When $x = 0$, then $z = -1$ for every value of y; when $y = 0$ then $z = +1$ for every value of x. In general, if $y = mx$,

$$z = \frac{1 + m}{1 - m},$$

and z may be made to have any value whatever at $(0, 0)$ by giving an appropriate value to m.

68. Partial differentiation. If in the function

$$z = f(x, y)$$

a fixed value y_1 be given to y, then

$$z = f(x, y_1)$$

is a function of x only, and the rate of change in z caused by a change in x is expressed by

$$dz = \frac{dz}{dx} dx, \tag{1}$$

in which $\frac{dz}{dx}$ is obtained on the supposition that y is constant.

To indicate this fact without the qualifying verbal statement, equation (1) will be written in the form

$$d_x z = \frac{\partial z}{\partial x} dx. \tag{2}$$

The symbol $\frac{\partial z}{\partial x}$ represents the result obtained by differentiating z with regard to x, the variable y being treated as a constant; it is called the *partial derivative* of z with regard to x.

From the definition of differentiation, Art. **11**, the partial derivative is the result of the indicated operation

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Similarly, the symbol $\frac{\partial z}{\partial y}$ represents the result obtained by differentiating z with regard to y, the variable x being treated as a constant; it is called the partial derivative of z with regard to y.

The partial derivative of z with regard to y is accordingly the result of the indicated operation

$$\frac{\partial z}{\partial y}=\lim_{\Delta y \doteq 0}\frac{f(x, y+\Delta y)-f(x, y)}{\Delta y}.$$

$d_x z=\frac{\partial z}{\partial x}dx$ is called the *partial x-differential* of z, and

$d_y z=\frac{\partial z}{\partial y}dy$ is called the *partial y-differential* of z.

Geometrically, the two equations

$$z=f(x, y), \quad y=y_1$$

define the curve of section of the surface $z=f(x, y)$ made by the plane $y=y_1$. The derivative $\frac{\partial z}{\partial x}$ defines the slope of the tangent line to this curve.

Similarly, when x has a given constant value, $x=x_1$, the partial derivative $\frac{\partial z}{\partial y}$ is the expression for the slope of the tangent to the curve cut from the surface $z=f(x, y)$ by the plane $x=x_1$.

The equations of these two tangent lines at the point (x_1, y_1, z_1) are

$$y=y_1,\ z-z_1=\frac{\partial z_1}{\partial x_1}(x-x_1),$$

$$x=x_1,\ z-z_1=\frac{\partial z_1}{\partial y_1}(y-y_1),$$

and hence the equation of the plane containing these two intersecting lines is

$$z-z_1=\frac{\partial z_1}{\partial x_1}(x-x_1)+\frac{\partial z_1}{\partial y_1}(y-y_1).$$

The plane is called the tangent plane to the surface $z=f(x, y)$ at the point (x_1, y_1, z_1).

EXERCISES

1. Given $u = x^4 + 3x^2y^2 - 7xy^3$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 4u.$

2. Given $u = \tan^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

3. $u = \log(e^x + e^y)$; find $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}.$

4. $u = \sin xy$; find $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}.$

5. $u = \log(x + \sqrt{x^2 + y^2})$; find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}.$

6. $u = \log(\tan x + \tan y + \tan z)$; show that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

7. $u = \log(x + y)$; show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = \dfrac{2}{e^u}.$

69. Total differential. If both x and y be allowed to vary in the function $z = f(x, y)$, the first question that naturally arises is to determine the meaning of the differential of z.

Let $$z_1 = f(x_1, y_1),$$

and $$z_1 + \Delta z = f(x_1 + \Delta x, y_1 + \Delta y)$$

be two values of the function corresponding to the two pairs of values of the variables x_1, y_1 and $x_1 + \Delta x$, $y_1 + \Delta y$.

The difference

$$\Delta z = f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1, y_1)$$

may be regarded as composed of two parts, the first part being the increment which z takes when x changes from x_1 to $x_1 + \Delta x$, while y remains constant $(y = y_1)$, and the second part being the additional increment which z takes when

y changes from y_1 to $y_1 + \Delta y$, while x remains constant $(x = x_1 + \Delta x)$. The increment Δz may then be written

$$\begin{aligned}\Delta z &= f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1 + \Delta x, y_1) \\ &\qquad + f(x_1 + \Delta x, y_1) - f(x_1, y_1) \\ &= \frac{f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1 + \Delta x, y_1)}{\Delta y} \Delta y \\ &\qquad + \frac{f(x_1 + \Delta x, y_1) - f(x_1, y_1)}{\Delta x} \Delta x.\end{aligned}$$

From the theorem of mean value, Art. 45, the last equation may be written

$$\Delta z = \frac{\partial}{\partial x} f(x_1 + \theta \Delta x, y_1) \Delta x + \frac{\partial}{\partial y} f(x_1 + \Delta x, y_1 + \theta_1 \Delta y) \Delta y.$$

It represents the actual increment Δz which the dependent variable z takes when the independent variables x and y take the increments Δx and Δy.

In the preceding equation let Δx, Δy, Δz be replaced by $\epsilon \cdot dx$, $\epsilon \cdot dy$, $\epsilon \cdot dz$ respectively, in which dx, dy are entirely arbitrary. After removing the common factor ϵ, let ϵ approach zero. The result is

$$dz = \frac{\partial f(x_1, y_1)}{\partial x} dx + \frac{\partial f(x_1, y_1)}{\partial y} dy. \tag{1}$$

The differential dz defined by this equation is called the *total differential* of z. It is not an actual increment of z, but the increment which z would take if the change continued uniform while x changed from x_1 to $x_1 + dx$ and y changed from y_1 to $y_1 + dy$. Geometrically speaking this is the increment which z would take if the point (x, y, z) should move from the position (x_1, y_1, z_1) in the tangent plane of the surface $z = f(x, y)$ instead of on the surface itself.

Equation (1) may be written in the form

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$$

from which the following theorem can be stated: *the total differential of a function of two variables is equal to the sum of the partial differentials.*

The same method can be directly applied to functions of three or more variables. Thus, if z be a function of the variables x, y, u,

$$z = \phi(x, y, u),$$

then $$dz = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial u}du.$$

Ex. **1.** Given $z = axy^2 + bx^2y + cx^3 + ey$,

then $dz = (ay^2 + 2\,bxy + 3\,cx^2)dx + (2\,axy + bx^2 + e)dy.$

Ex. **2.** Given $z = x^y$, then $d_xz = yx^{y-1}dx$, and $d_yz = x^y \log x\,dy$.

Hence $dz = yx^{y-1}dx + x^y \log x\,dy.$

Ex. **3.** Given $u = \tan^{-1}\frac{y}{x}$, show that $du = \frac{x\,dy - y\,dx}{x^2 + y^2}$.

Ex. **4.** Assuming the characteristic equation of a perfect gas, $vp = Rt$, in which v is volume, p pressure, t absolute temperature, and R a constant, express each of the differentials dv, dp, dt, in terms of the other two.

Ex. **5.** A particle moves on the spherical surface $x^2 + y^2 + z^2 = a^2$ in a vertical meridian plane inclined at an angle of 60° to the zx plane.

If the x-component of its velocity be $\frac{1}{10}a$ per second when $x = \frac{1}{4}a$, find the y-component and the z-component velocity.

Since $$z = \sqrt{a^2 - x^2 - y^2},$$

then $$dz = -\frac{x\,dx}{\sqrt{a^2 - x^2 - y^2}} - \frac{y\,dy}{\sqrt{a^2 - x^2 - y^2}}.$$

But since $dx = \frac{1}{10}a$, and the equation of the given meridian plane is $y = x\tan 60°$, hence $dy = dx\sqrt{3} = \frac{a}{10}\sqrt{3}$, and $y = \frac{a\sqrt{3}}{4}$. Therefore

$$dz = -\frac{dx}{2\sqrt{3}} - \frac{dy}{2} = -\frac{a\sqrt{3}}{15} \text{ in feet per second.}$$

70. Language of differentials. The results of the preceding articles may be stated thus:

The partial z-differential due to a change in x is equal to the x-differential multiplied by the partial x-derivative.

The partial z-differential due to a change in y is equal to the y-differential multiplied by the partial y-derivative.

The total z-differential is equal to the sum of the partial z-differentials.

One advantage of writing the equation in the differential form is that it may be divided when necessary by the differential of any other variable s, to which x and y may be related, and then, remembering that the ratio of two differentials (or rates) may be expressed as a derivative, the equation would become

$$\frac{dz}{ds}=\frac{\partial z}{\partial x}\frac{dx}{ds}+\frac{\partial z}{\partial y}\frac{dy}{ds}.$$

In particular, if y be not independent, but is a function of x, then s may be chosen as x itself, and the preceding equation becomes

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}.$$

If the functional relation between x and y be given,

$$y=\phi(x),$$

then the same result would be obtained, whether $\frac{dz}{dx}$ be determined by the present method, or y be first eliminated from the relation

$$z=f(x, y),$$

and the resulting equation be differentiated as to x. The method of this article frequently shortens the process.

It is here well to note the difference between $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$. The former is the partial derivative of the functional expression for z with regard to x, on the supposition that y is constant. The latter is the total derivative of z with regard to x, when account is taken of the fact that y is itself a function of x.

Ex. 1. Given $z=\sqrt{x^2+y^2}$, $y=\log x$; find $\frac{dz}{dx}$.

$$\frac{dz}{dx}=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dx},$$

$$\frac{dy}{dx}=\frac{1}{x},$$

hence $$\frac{dz}{dx}=\frac{x^2+y}{x\sqrt{x^2+y^2}}.$$

Ex. 2. If $z=\tan^{-1}\frac{y}{2x}$ and $4x^2+y^2=1$, show that $\frac{dz}{dx}=\frac{-2}{y}$.

71. Differentiation of implicit functions. If in the relation $z=f(x, y)$, z be assumed to be constant, then

$$dz=0;$$

hence $$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0, \qquad (1)$$

from which $$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \qquad (2)$$

In all such cases either variable is an implicit function of the other, and thus the last equation furnishes a rule for finding the derivative of an implicit function.

Ex. 1. Given $x^3+y^3+3axy=c$, find $\frac{dy}{dx}$.

Since $(3x^2+3ay)+(3y^2+3ax)\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{x^2+ay}{y^2+ax}$.

Ex. 2. $f(ax+by)=c$; $\frac{\partial f}{\partial x}=af'(ax+by)$; $\frac{\partial f}{\partial y}=bf'(ax+by)$; $\frac{dy}{dx}=-\frac{a}{b}$

Ex. 3. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, find $\frac{dy}{dx}$.

Ex. 4. Given $x^4-y^4=c$, find $\frac{dy}{dx}$.

It is to be noticed that the result of differentiating any implicit function of x, y by the method of the present article agrees with the result of differentiation according to the rules of Chapter II. Compare Ex. 2, p. 53.

72. Successive partial differentiation. The expressions $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ which were defined in Art. 68 are functions of both x and y.

If $\frac{\partial z}{\partial x}$ be differentiated partially as to x, the result is written

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}.$$

This expression is called the second partial derivative of z as to x.

Similarly, the results of the operations indicated by

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),\quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

are written $\frac{\partial^2 z}{\partial y\,\partial x}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y^2}$, respectively.

Beginning with the left, these expressions are called the second partial derivative of z as to x and y, the second partial derivative of z as to y and x, and the second partial derivative of z as to y.

73. Order of differentiation indifferent.

Theorem. The successive partial derivatives

$$\frac{\partial^2 z}{\partial y\,\partial x},\quad \frac{\partial^2 z}{\partial x\,\partial y}$$

are equal for any values of x and y in the vicinity of which z and its first and second partial x- and y-derivatives are continuous.

For, in $z=f(x, y)$, first change x into $x+h$, keeping y

constant. Then by the theorem of mean value (Art. 45), the increment of the function is equal to the increment of the variable multiplied by the derivative taken for some value intermediate between x and $x + h$; that is,

$$f(x+h, y) - f(x, y) = h\frac{\partial}{\partial x}f(x+\theta h, y). \qquad [0 < \theta < 1.$$

Next let y change to $y + k$, x remaining constant, and take the increment of the function on the left. Then by the theorem of mean value applied to $\frac{\partial}{\partial x}f(x+\theta h, y)$ as a function of y with the increment k,

$$[f(x+h, y+k) - f(x, y+k)] - [f(x+h, y) - f(x, y)]$$
$$= kh\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x+\theta h, y+\theta_1 k).$$

Now let these increments be given in reversed order. Then

$$[f(x+h, y+k) - f(x+h, y)] - [f(x, y+k) - f(x, y)]$$
$$= hk\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x+\theta_2 h, y+\theta_3 k);$$

hence

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x+\theta h, y+\theta_1 k) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x+\theta_2 h, y+\theta_3 k).$$

This relation is true for any values of h and k for which all the functions mentioned are continuous.

When h, k approach zero,

$$x+\theta h,\ y+\theta_1 k, \text{ and } x+\theta_2 h,\ y+\theta_3 k$$

approach x, y, and

$$f(x+\theta h, y+\theta_1 k),\ f(x+\theta_2 h, y+\theta_3 k)$$

approach $f(x, y)$; and similarly for the derivatives; hence

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x, y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x, y),$$

or, since $f(x, y) = z$,

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Cor. It follows directly that under corresponding conditions the order of differentiation in the higher partial derivatives is indifferent.

E.g.,
$$\frac{\partial^3 z}{\partial x\,\partial y\,\partial x}=\frac{\partial^3 z}{\partial x^2\,\partial y}=\frac{\partial^3 z}{\partial y\,\partial x^2}.$$

EXERCISES

✓ **1.** Verify that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$, when $u=x^2y^3$.

✓ **2.** Verify that $\dfrac{\partial^3 u}{\partial x\,\partial y^2}=\dfrac{\partial^3 u}{\partial y^2\,\partial x}$, when $u=x^2y+xy^3$.

✓ **3.** Verify that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$, when $u=y\log(1+xy)$.

4. In Ex. 3 are there any exceptional values of x, y for which the relation is not true?

✓ **5.** Given $u=(x^2+y^2)^{\frac{1}{2}}$, verify the formula
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

✓ **6.** Given $u=(x^3+y^3)^{\frac{1}{3}}$, show that the expression in the left member of the differential equation in Ex. 5 is equal to $\dfrac{3u}{4}$.

✓ **7.** Given $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$; prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$.

8. Given $u=\sec(y+ax)+\tan(y-ax)$; prove that $\dfrac{\partial^2 u}{\partial x^2}=a^2\dfrac{\partial^2 u}{\partial y^2}$.

✓ **9.** Given $u=\sin x\cos y$; verify that $\dfrac{\partial^4 u}{\partial y^2\,\partial x^2}=\dfrac{\partial^4 u}{\partial x\,\partial y\,\partial x\,\partial y}=\dfrac{\partial^4 u}{\partial x^2\,\partial y^2}$.

✓ **10.** Given $u=(4ab-c^2)^{-\frac{1}{2}}$; prove that $\dfrac{\partial^2 u}{\partial c^2}=\dfrac{\partial^2 u}{\partial a\,\partial b}$.

CHAPTER IX

CHANGE OF VARIABLE

74. Interchange of dependent and independent variables. If y be a continuous function of x, defined by the equation $f(x, y) = 0$, the symbol $\frac{dy}{dx}$ represents the derivative of y with regard to x, when one exists. If x be regarded as a function of y, defined by the same equation, the symbol $\frac{dx}{dy}$ represents the derivative of x with regard to y, when one exists. It is required to find the relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$.

Let x, y change from the initial values x_1, y_1 to the values $x_1 + \Delta x$, $y_1 + \Delta y$, subject to the relation $f(x, y) = 0$.

Then, since

$$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}},$$

it follows, by taking the limit, that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}. \tag{1}$$

Hence, *if y and x be connected by a functional relation the derivative of y with regard to x is the reciprocal of the derivative of x with regard to y.*

This process is known as changing the independent variable from x to y. The corresponding relations for the higher derivatives are less simple. They are obtained in the following manner:

To express $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, differentiate (1) as to x;

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{1}{\frac{dx}{dy}}\right]=\frac{d}{dy}\left[\frac{1}{\frac{dx}{dy}}\right]\cdot\frac{dy}{dx}=\frac{d}{dy}\left[\frac{1}{\frac{dx}{dy}}\right]\cdot\frac{1}{\frac{dx}{dy}}.$$

But
$$\frac{d}{dy}\left[\frac{1}{\frac{dx}{dy}}\right]=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2},$$

hence
$$\frac{d^2y}{dx^2}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}. \tag{2}$$

In a similar manner,

$$\frac{d^3y}{dx^3}=-\frac{\frac{d^3x}{dy^3}\frac{dx}{dy}-3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5}. \tag{3}$$

75. Change of the dependent variable. If y is a function of z, let it be required to express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\cdots$ in terms of $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, $\cdots$.

Suppose $y=\phi(z)$. Then

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\phi'(z)\frac{dz}{dx},$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\phi'(z)\frac{dz}{dx}\right)$$

$$=\frac{dz}{dx}\frac{d}{dx}\phi'(z)+\phi'(z)\frac{d^2z}{dx^2}.$$

But $$\frac{d}{dx}\phi'(z) = \frac{d}{dz}\phi'(z)\frac{dz}{dx} = \phi''(z)\frac{dz}{dx}.$$

Hence $$\frac{d^2y}{dx^2} = \phi''(z)\left(\frac{dz}{dx}\right)^2 + \phi'(z)\frac{d^2z}{dx^2}. \qquad (4)$$

The higher x-derivatives of y can be similarly expressed in terms of x-derivatives of z.

76. Change of the independent variable. Let y be a function of x, and let both x and y be functions of a new variable t. It is required to express $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$, and $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

By Art. 14,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad (1)$$

hence $$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}. \qquad (2)$$

In practical examples it is usually better to work by the methods here illustrated than to use the resulting formulas.

EXERCISES

1. Change the independent variable from x to z in the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0, \text{ when } x = e^z.$$

$$\frac{dy}{dx} = \frac{dy}{dz}e^{-z},$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}e^{-2z} - \frac{dy}{dz}e^{-2z}.$$

Hence $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ becomes } \frac{d^2y}{dz^2} + y = 0.$$

✓ **2.** Interchange the function and the variable in the equation

$$\frac{d^2y}{dx^2}+2\,y\left(\frac{dy}{dx}\right)^2=0.$$

✓ **3.** Interchange x and y in the equation

$$R=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

✓ **4.** Change the independent variable from x to y in the equation

$$3\left(\frac{d^2y}{dx^2}\right)^2-\frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2=0.$$

+ **5.** Change the dependent variable from y to z in the equation

$$\frac{d^2y}{dx^2}=1+\frac{2\,(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2,\quad\text{when}\quad y=\tan z.$$

✓ **6.** Change the independent variable from x to y in the equation

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+u=0,\quad\text{when}\quad y=\log x.$$

7. If y is a function of x, and x a function of the time t, express the y-acceleration in terms of the x-acceleration, and the x-velocity.

Since $$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt},$$

hence $$\frac{d^2y}{dt^2}=\frac{dy}{dx}\frac{d^2x}{dt^2}+\frac{dx}{dt}\cdot\frac{d}{dt}\left(\frac{dy}{dx}\right).$$

But $$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}=\frac{d^2y}{dx^2}\frac{dx}{dt},$$

hence $$\frac{d^2y}{dt^2}=\frac{dy}{dx}\frac{d^2x}{dt^2}+\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2.$$

In the abbreviated notation for t-derivatives,

$$d^2y=\frac{dy}{dx}d^2x+\frac{d^2y}{dx^2}(dx)^2.$$

+ **8.** Change the independent variable from x to u in the equation

$$\frac{d^2y}{dx^2}+\frac{2\,x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0,\quad\text{when}\quad x=\tan u.$$

9. Change the independent variable from x to t in the equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0, \quad\text{when}\quad x=\cos t.$$

10. Show that the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

remains unchanged in form by the substitution $x=\dfrac{1}{z}$.

✓ **11.** Interchange the variable and the function in the equation

$$\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-y\left(\frac{dy}{dx}\right)^3=0.$$

12. Change the dependent variable from y to z in the equation

$$\frac{d^2y}{dx^2}+(1-y)\frac{dy}{dx}+y^2=0, \quad\text{when}\quad y=z^2.$$

APPLICATIONS TO GEOMETRY

CHAPTER X

TANGENTS AND NORMALS

77. It was shown in Art. 10 that if $f(x, y) = 0$ be the equation of a plane curve, then $\frac{dy}{dx}$ measures the slope of the tangent to the curve at the point x, y. The slope at a particular point (x_1, y_1) will be denoted by $\frac{dy_1}{dx_1}$, meaning that x_1 is to be substituted for x, and y_1 for y in the expression for $\frac{dy}{dx}$.

78. Equation of tangent and normal at a given point. Since the tangent line passes through the given point (x_1, y_1) and has the slope $\frac{dy_1}{dx_1}$, its equation is

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1). \tag{1}$$

The *normal* to the curve at the point (x_1, y_1) is the straight line through this point, perpendicular to the tangent.

Since the slope of the normal is

$$\frac{-1}{\frac{dy}{dx}} = -\frac{dx}{dy}, \qquad \text{[Art. 74,}$$

its equation is $$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1),$$

i.e., $$(x - x_1) + \frac{dy_1}{dx_1}(y - y_1) = 0. \tag{2}$$

79. Length of tangent, normal, subtangent, subnormal.

The segments of the tangent and normal intercepted between the point of tangency and the axis OX are called, respectively, the *tangent length* and the *normal length,* and their projections on OX are called the *subtangent* and the *subnormal.*

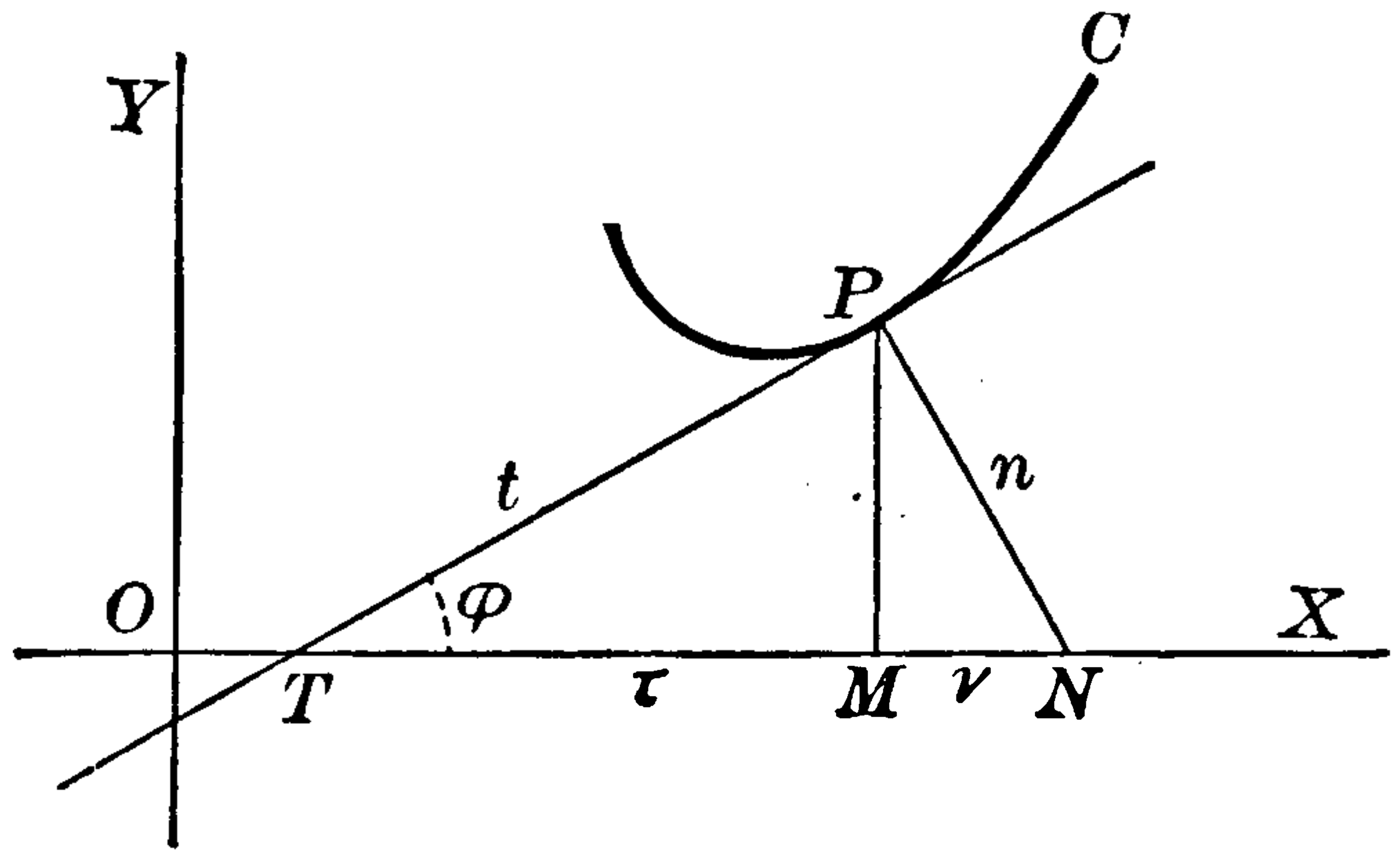


FIG. 23 *a*.

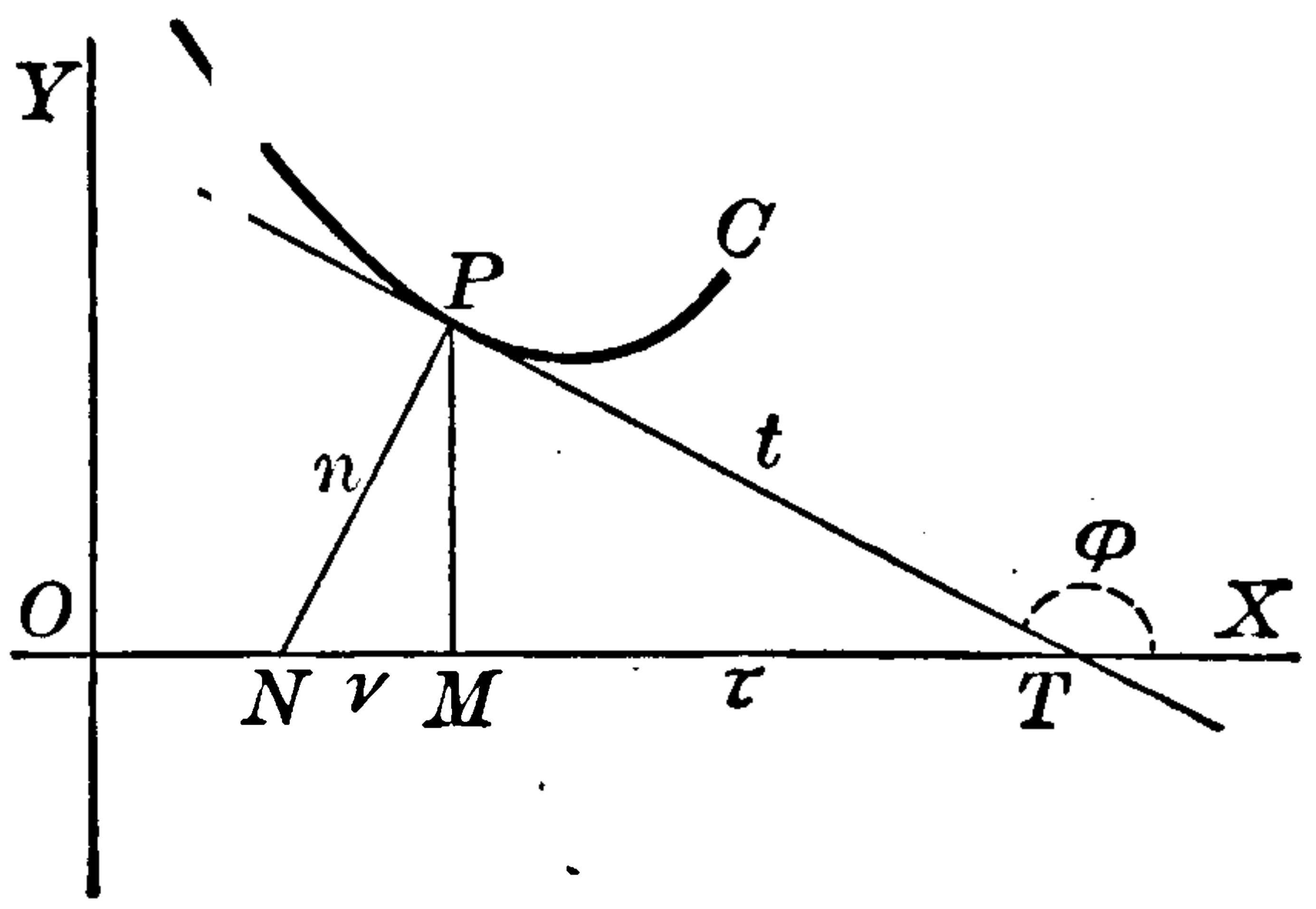


FIG. 23 *b*.

Thus, in Fig. 23, let the tangent and normal to the curve PC at P meet the axis OX in T and N, and let MP be the ordinate of P. Then

TP is the tangent length,
PN the normal length,
TM the subtangent,
MN the subnormal.

These will be denoted, respectively, by t, n, τ, ν.

Let the angle XTP be denoted by ϕ, and write $\tan\phi = m$.

Then
$$\cos\phi = \frac{1}{\sqrt{1+m^2}};\quad \sin\phi = \frac{m}{\sqrt{1+m^2}};$$

$$t = \frac{y_1}{\sin\phi} = \frac{y_1\sqrt{1+m^2}}{m};\quad n = \frac{y_1}{\cos\phi} = y_1\sqrt{1+m^2};$$

$$\tau = y_1\cot\phi = y_1\frac{dx_1}{dy_1} = \frac{y_1}{m};\quad \nu = y_1\tan\phi = y_1\frac{dy_1}{dx_1} = my_1.$$

The subtangent is measured from the intersection of the tangent to the foot of the ordinate; it is therefore positive when the foot of the ordinate is to the right of the intersection of tangent. The subnormal is measured from the foot of the ordinate to the intersection of normal, and is positive when the normal cuts OX to the right of the foot of the ordinate. Both are therefore positive or negative, according as ϕ is acute or obtuse.

The expressions for τ, ν may also be obtained by finding from equations (1), (2), Art. 78, the intercepts made by the tangent and normal on the axis OX. The intercept of the tangent subtracted from x_1 gives τ, and x_1 subtracted from the intercept of the normal gives ν.

Ex. Find the intercepts made upon the axes by the tangent at the point (x_1, y_1) on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and show that their sum is constant.

Differentiating the equation of the curve,

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} \frac{dy}{dx} = 0.$$

Hence the equation of the tangent is

$$y - y_1 = -\sqrt{\frac{y_1}{x_1}}(x - x_1).$$

The x intercept is $x_1 + \sqrt{x_1 y_1}$, and the y intercept is $y_1 + \sqrt{x_1, y_1}$, hence their sum is

$$(\sqrt{x_1} + \sqrt{y_1})^2 = a.$$

If a series of lines be drawn such that the sum of the intercepts of each is the same constant, account being taken of the signs, the form of the parabola to which they are all tangent can be readily seen.

EXERCISES

1. Find the equations of the tangent and the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1). Compare the process with that employed in analytic geometry to obtain the same results.

✓ **2.** Find the equation of the tangent to the curve $x^2(x+y)=a^2(x-y)$ at the origin.

✓ **3.** Find the equations of the tangent and normal at the point (1, 3) on the curve $y^2=9x^3$.

✓ **4.** Find the equations of the tangent and normal to each of the following curves at the point indicated:

(α) $y=\dfrac{8a^3}{4a^2+x^2}$, at the point for which $x=2a$.

(β) $y^2=2x^2-x^3$, at the points for which $x=1$.

(γ) $y^2=4px$, at the point $(p, 2p)$.

✓ **5.** Find the value of the subtangent of $y^2=3x^2-12$ at $x=4$. Compare the process with that already given in analytic geometry.

✓ **6.** Find the length of the tangent to the curve $y^2=2x$ at $x=8$.

7. Find the points at which the tangent is parallel to the axis of x, and at which it is perpendicular to the x axis for each of the following curves:

(α) $ax^2+2hxy+by^2=1$.

(β) $y=\dfrac{x^3-a^3}{ax}$.

(γ) $y^3=x^2(2a-x)$.

8. Find the condition that the conics $ax^2+by^2=1$, $a'x^2+b'y^2=1$ shall cut at right angles.

9. Find the angle at which $x^2=y^2+5$ intersects $8x^2+18y^2=144$. Compare with Ex. 8.

10. Show that in the equilateral hyperbola $2xy=a^2$ the area of the triangle formed by a variable tangent and the coördinate axes is constant and equal to a^2.

11. At what angle does $y^2=8x$ intersect $4x^2+2y^2=48$?

✓ **12.** Determine the subnormal to the curve $y^n=a^{n-1}x$.

13. Find the values of x for which the tangent to the curve

$$y^3=(x-a)^2(x-c)$$

is parallel to the axis of x.

14. Show that the subtangent of the hyperbola $xy=a^2$ is equal to the abscissa of the point of tangency, but opposite in sign.

✓ **15.** Prove that the parabola $y^2=4ax$ has a constant subnormal.

16. Show analytically that in the curve $x^2 + y^2 = a^2$ the length of the normal is constant.

17. Show that in the tractrix, the length of the tangent is constant, the equation of the tractrix being

$$x = \sqrt{c^2 - y^2} + \frac{c}{2} \log \frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 - y^2}}.$$

18. Show that the exponential curve $y = ae^{\frac{x}{c}}$ has a constant subtangent.

19. Find the point on the parabola $y^2 = 4px$ at which the angle between the tangent and the line joining the point to the vertex shall be a maximum.

POLAR COÖRDINATES

80. When the equation of a curve is expressed in polar coördinates, the vectorial angle θ is usually regarded as the independent variable. To determine the direction of the curve at any point, it is most convenient to make use of the angle between the tangent and the radius vector to the point of tangency.

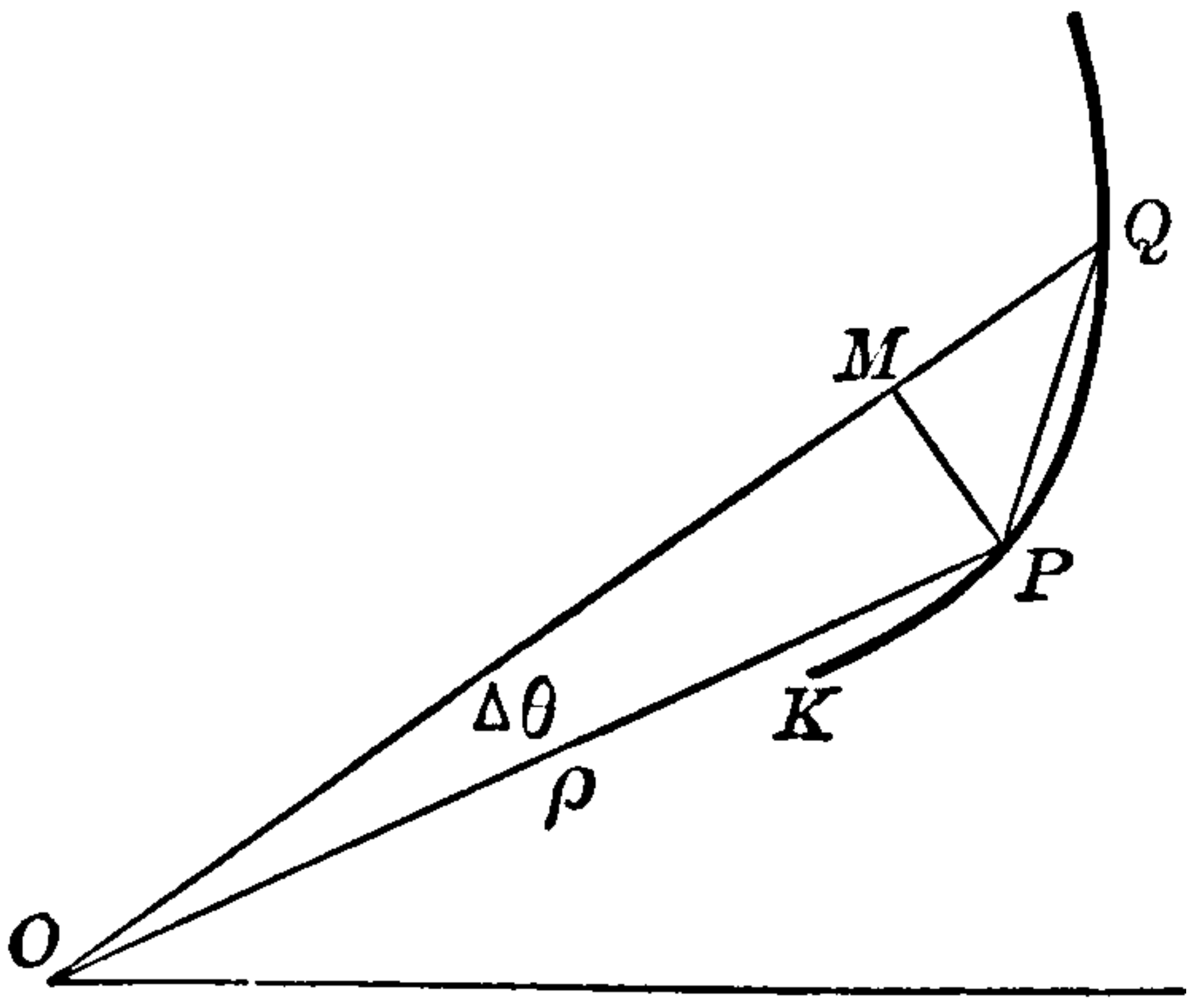


Fig. 24.

Let P, Q be two points on the curve (Fig. 24). Join P, Q with the pole O, and drop a perpendicular PM from P on OQ. Let ρ, θ be the coördinates of P; $\rho + \Delta\rho$, $\theta + \Delta\theta$ those of Q. Then the angle $POQ = \Delta\theta$; $PM = \rho \sin \Delta\theta$; and $MQ = OQ - OM = \rho + \Delta\rho - \rho \cos \Delta\theta$.

Hence $$\tan MQP = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}.$$

When Q moves to coincidence with P, the angle MQP approaches as a limit the angle between the radius vector and the tangent line at the point P. This angle will be designated by ψ.

Thus $$\tan\psi = \lim_{\Delta\theta \doteq 0} \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho\cos\Delta\theta}.$$

But $\rho(1 - \cos\Delta\theta) = 2\rho\sin^2 \frac{1}{2}\Delta\theta,$

hence $$\tan\psi = \lim_{\Delta\theta \doteq 0} \frac{\dfrac{\rho\sin\Delta\theta}{\Delta\theta}}{\rho\sin\frac{1}{2}\Delta\theta \cdot \dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta} + \dfrac{\Delta\rho}{\Delta\theta}}.$$

Since $\lim\limits_{\Delta\theta \doteq 0} \dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta} = 1$, the preceding equation reduces to

$$\tan\psi = \frac{\rho}{\dfrac{d\rho}{d\theta}} = \rho\frac{d\theta}{d\rho}.$$

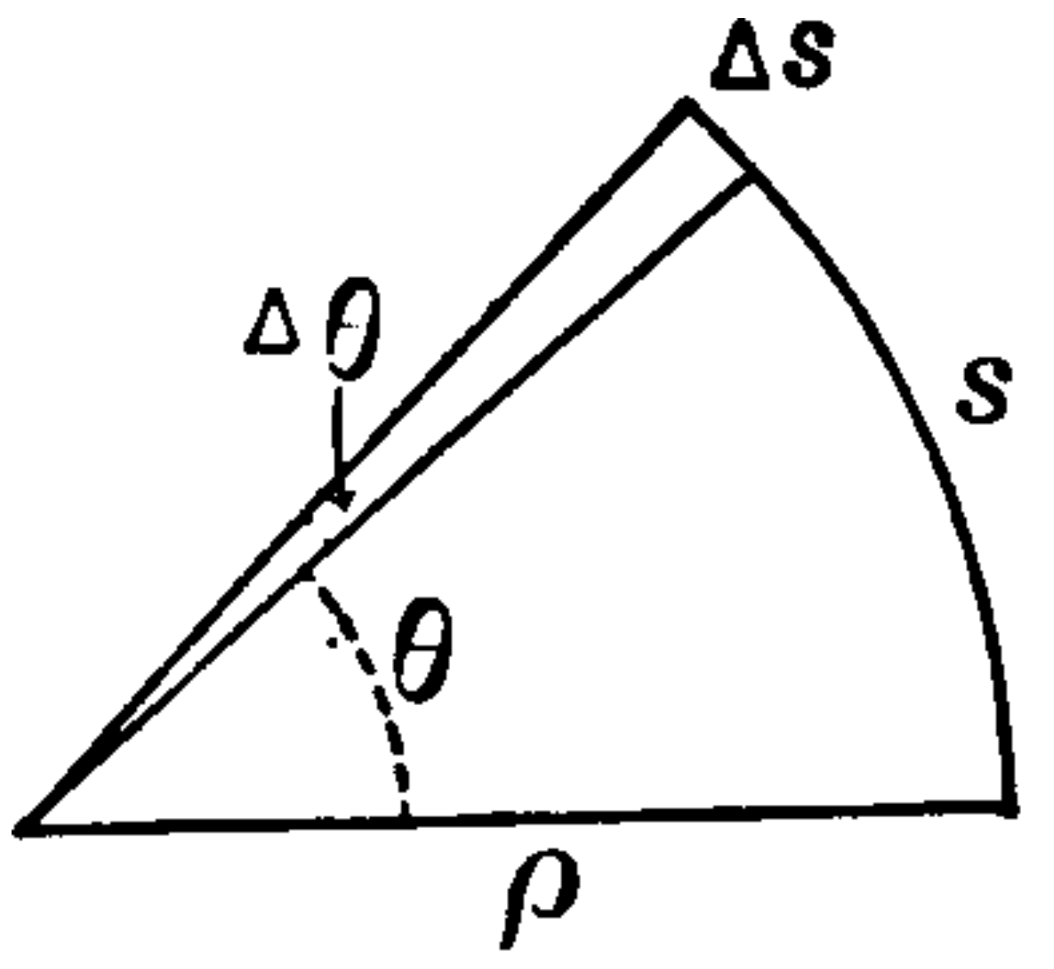


FIG. 25.

Ex. 1. A point describes a circle of radius ρ. Prove that at any instant the arc velocity is ρ times the angle velocity,

i.e., $$\frac{ds}{dt} = \rho\frac{d\theta}{dt}.$$

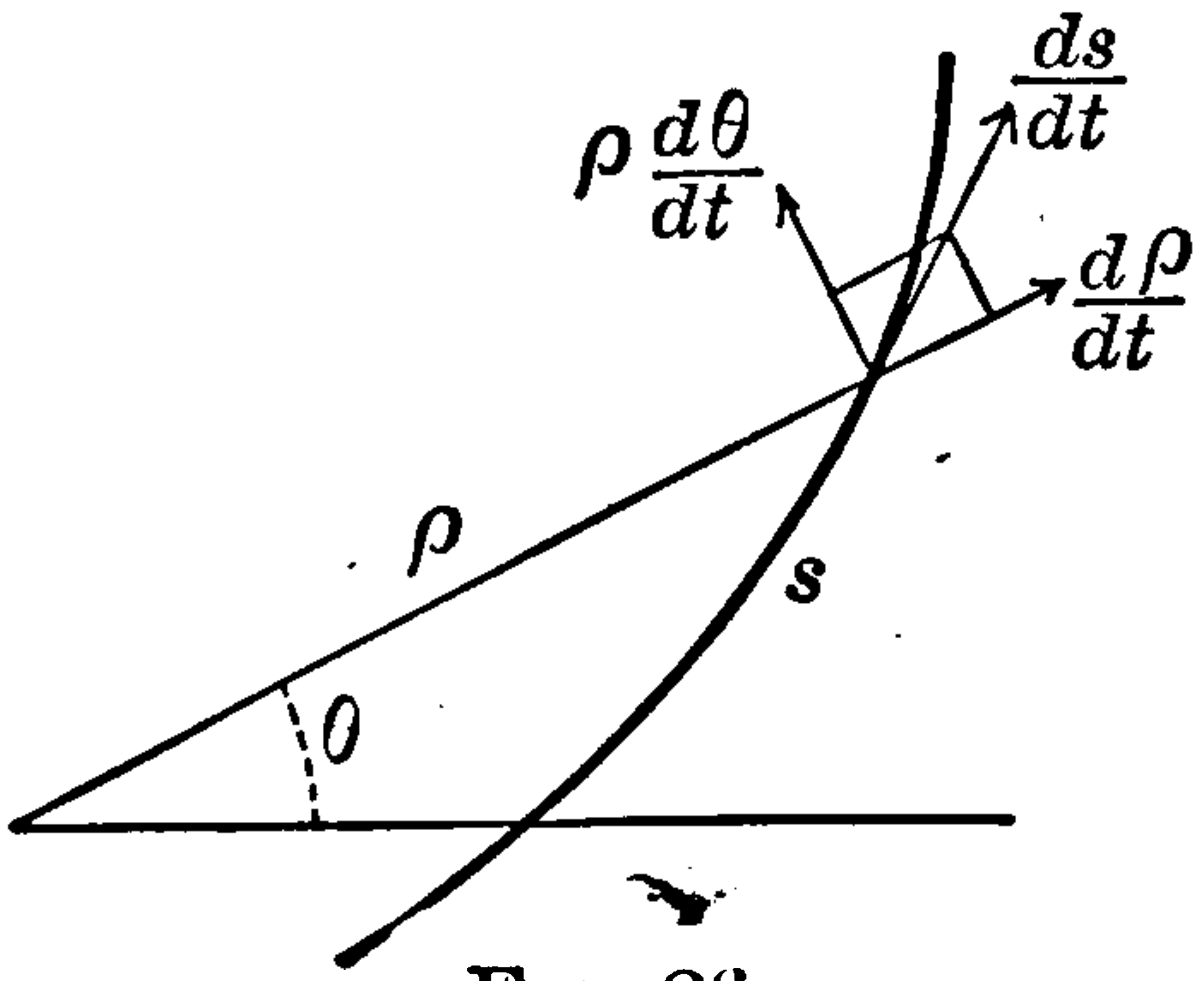


FIG. 26.

Ex. 2. When a point describes a given curve, prove that at any instant the velocity $\frac{ds}{dt}$ has a radius component $\frac{d\rho}{dt}$ and a component perpendicular to the radius vector $\rho\frac{d\theta}{dt}$, and hence that

$$\cos\psi = \frac{d\rho}{ds},\ \sin\psi = \rho\frac{d\theta}{ds},\ \tan\psi = \rho\frac{d\theta}{d\rho}.$$

81. Relation between $\frac{dy}{dx}$ and $\frac{\rho d\theta}{d\rho}$. If the initial line be taken as the axis of x, the tangent line at P makes an angle ϕ with this line.

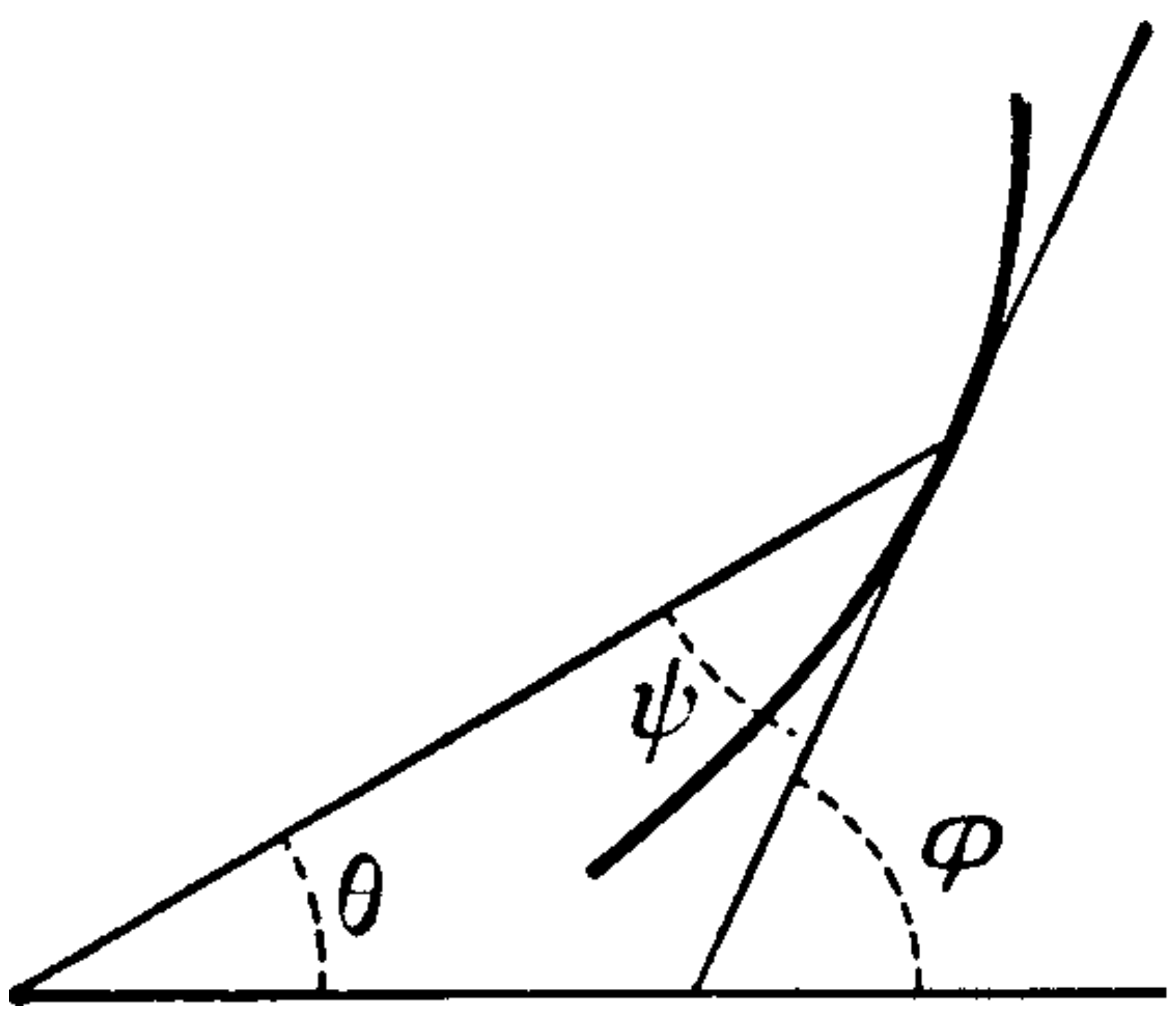


FIG. 27.

Hence $$\theta + \psi = \phi;$$

i.e., $$\theta + \tan^{-1}\left(\rho\frac{d\theta}{d\rho}\right) = \tan^{-1}\left(\frac{dy}{dx}\right).$$

82. Length of tangent, normal, polar subtangent, and polar subnormal. The portions of the tangent and normal intercepted between the point of tangency P and the line through the pole perpendicular to the radius vector OP, are called the *polar tangent length* and the *polar normal length;* their projections on this perpendicular are called the *polar subtangent* and *polar subnormal.*

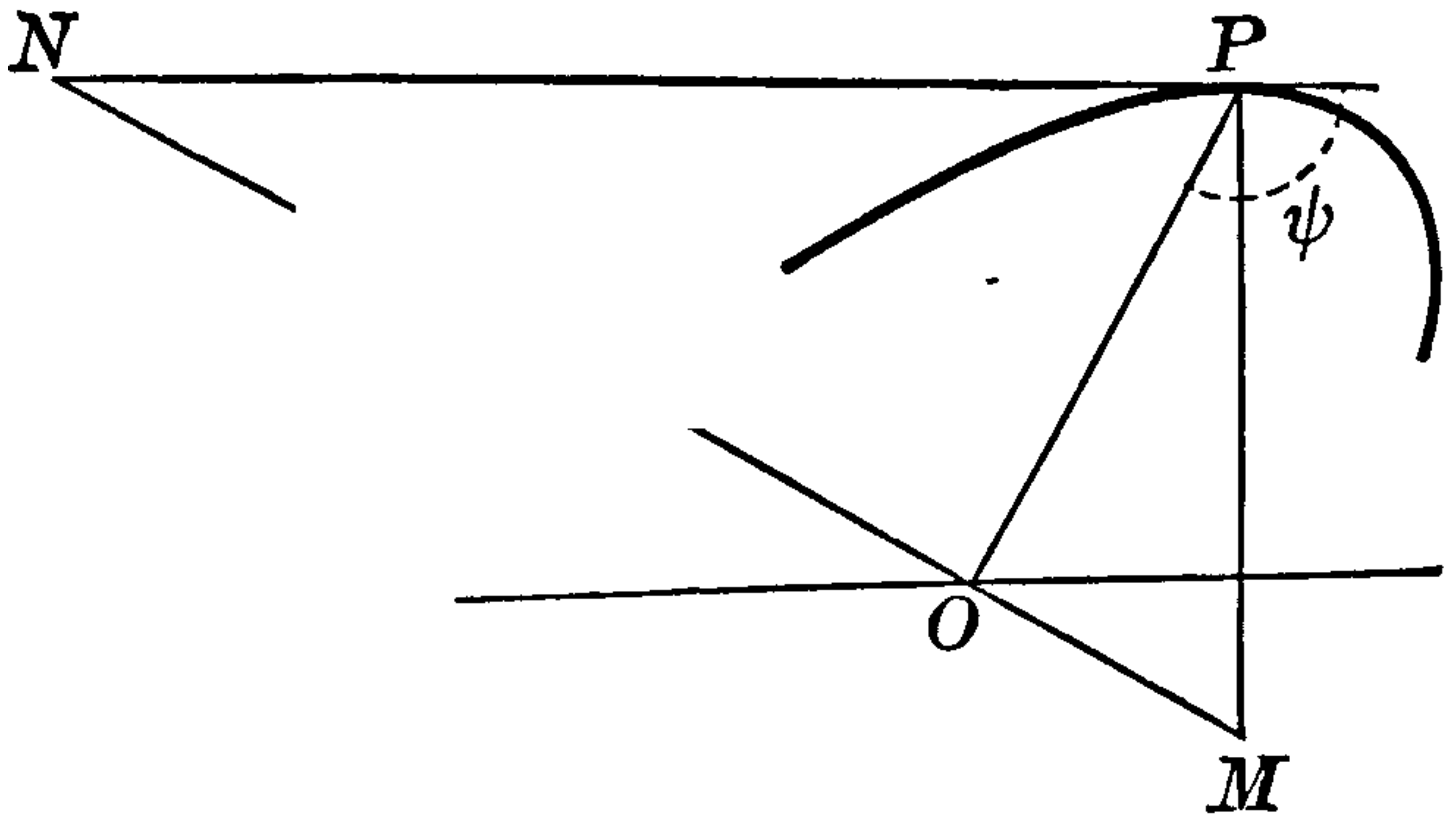


FIG. 28 *a.*

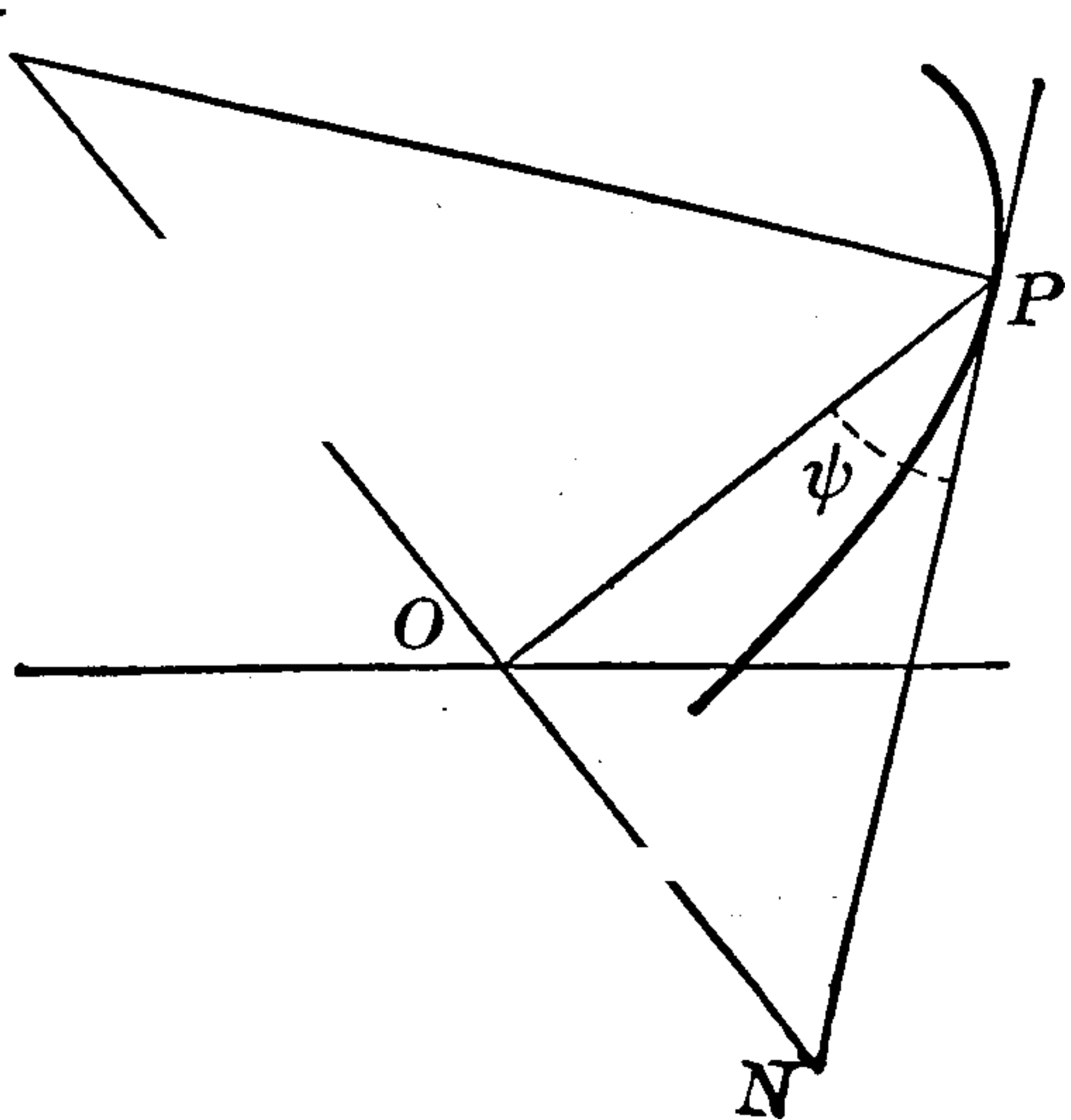


FIG. 28 *b.*

Thus, let the tangent and normal at P meet the perpendicular to OP in the points N and M. Then

PN is the polar tangent length,
PM is the polar normal length,
ON is the polar subtangent,
OM is the polar subnormal.

They are all seen to be independent of the direction of the initial line. The lengths of these lines will now be determined.

Since $PN = OP\cdot\sec OPN = \rho\sec\psi = \rho\sqrt{\rho^2\left(\dfrac{d\theta}{d\rho}\right)^2+1}$

$$= \rho\frac{d\theta}{d\rho}\sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2},$$

hence polar tangent length $= \rho\dfrac{d\theta}{d\rho}\sqrt{\rho^2+\left(\dfrac{d\rho}{d\theta}\right)^2}$.

Again, $ON = OP \tan OPN = \rho \tan \psi = \rho^2 \dfrac{d\theta}{d\rho}$,

hence $$\text{polar subtangent} = \rho^2 \frac{d\theta}{d\rho}.$$

$$PM = OP \cdot \csc OPN = \rho \csc \psi = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2},$$

hence $$\text{polar normal length} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}.$$

$$OM = OP \cot OPN = \frac{d\rho}{d\theta},$$

hence $$\text{polar subnormal} = \frac{d\rho}{d\theta}.$$

The signs of the polar tangent length and polar normal length are ambiguous on account of the radical. The direction of the subtangent is determined by the sign of $\rho^2 \dfrac{d\theta}{d\rho}$. When $\dfrac{d\theta}{d\rho}$ is positive, the distance ON should be measured to the right, and when negative, to the left of an observer placed at O and looking along OP; for when θ increases with ρ, $\dfrac{d\theta}{d\rho}$ is positive (Art. 13), and ψ is an acute angle (as in Fig. 28 *b*); when θ decreases as ρ increases, $\dfrac{d\theta}{d\rho}$ is negative, and ψ is obtuse (Fig. 28 *a*).

EXERCISES

1. In the curve $\rho = a \sin \theta$, find ψ.

2. In the spiral of Archimedes $\rho = a\theta$, show that $\tan \psi = \theta$ and find the polar subtangent, polar normal, and polar subnormal. Trace the curve.

3. Find for the curve $\rho^2 = a^2 \cos 2\theta$ the values of all the expressions treated in this article.

4. Show that in the curve $\rho\theta = a$ the polar subtangent is of constant length. Trace the curve.

5. In the curve $\rho = a(1 - \cos\theta)$, find ψ and the polar subtangent.

6. Show that in the curve $\rho = b \cdot e^{\theta \cot \alpha}$ the tangent makes a constant angle α with the radius vector. For this reason, this curve is called the equiangular spiral.

7. Find the angle of intersection of the curves

$$\rho = a(1 + \cos\theta), \quad \rho = b(1 - \cos\theta).$$

8. In the parabola $\rho = a\sec^2\frac{\theta}{2}$, show that $\phi + \psi = \pi$.

CHAPTER XI

DERIVATIVE OF AN ARC, AREA, VOLUME, AND SURFACE OF REVOLUTION

83. Derivative of an arc. The length s of the arc AP of a given curve $y = f(x)$, measured from a fixed point A to any point P, is a function of the abscissa x of the latter point, and may be expressed by a relation of the form $s = \phi(x)$.

The determination of the function ϕ when the form of f is known, is an important and sometimes difficult problem in the Integral Calculus. The first step in its solution is to determine the form of the derivative function $\frac{ds}{dx} = \phi'(x)$, which is easily done by the methods of the Differential Calculus.

Let PQ be two points on the curve (Fig. 29); let x, y be the coördinates of P; $x + \Delta x$, $y + \Delta y$ those of Q; s the length of the arc AP; $s + \Delta s$ that of the arc AQ. Draw the ordinates MP, NQ; and draw PR parallel to MN. Then $PR = \Delta x$, $RQ = \Delta y$; arc $PQ = \Delta s$. Hence

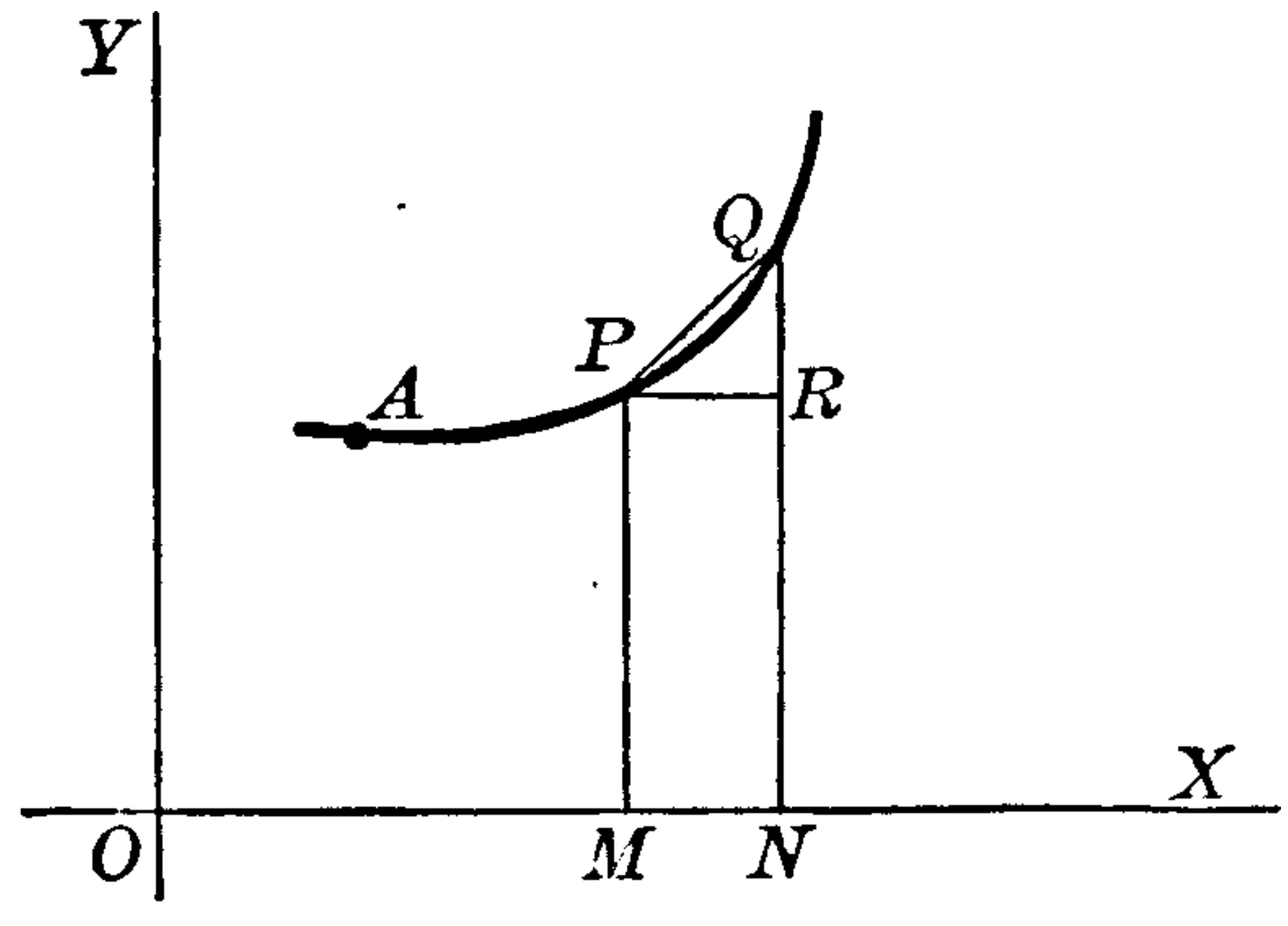


FIG. 29.

$$\text{Chord } PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

$$\frac{PQ}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Therefore $$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{PQ} \cdot \frac{PQ}{\Delta x} = \frac{\Delta s}{PQ}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Taking the limit of both members as Δx approaches zero and putting $\lim_{\Delta x \doteq 0} \frac{\Delta s}{PQ} = 1$, by Art. 6, Th. 4, and Art. 4, Th. 8, Cor., it follows that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{1}$$

Similarly $$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{2}$$

Moreover, from Art. 65

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \tag{3}$$

or in the differential notation

$$ds^2 = dx^2 + dy^2. \tag{4}$$

84. Trigonometric meaning of $\frac{ds}{dx}$, $\frac{ds}{dy}$.

Since $$\frac{\Delta x}{\Delta s} = \frac{\Delta x}{PQ} \cdot \frac{PQ}{\Delta s} = \cos RPQ \cdot \frac{PQ}{\Delta s},$$

it follows by taking the limit that

$$\frac{dx}{ds} = \cos \phi,$$

wherein ϕ, being the limit of the angle RPQ, is the angle which the tangent at the point (x, y) makes with the x-axis.

Similarly, $\frac{dy}{ds} = \sin \phi$; whence $\frac{ds}{dx} = \sec \phi$, $\frac{ds}{dy} = \csc \phi$.

Using the idea of a rate or differential, all these relations may be conveniently exhibited by Fig. 30.

These results may also be derived from equations (1), (2) of Art. 83, by putting $\frac{dy}{dx} = \tan \phi$.

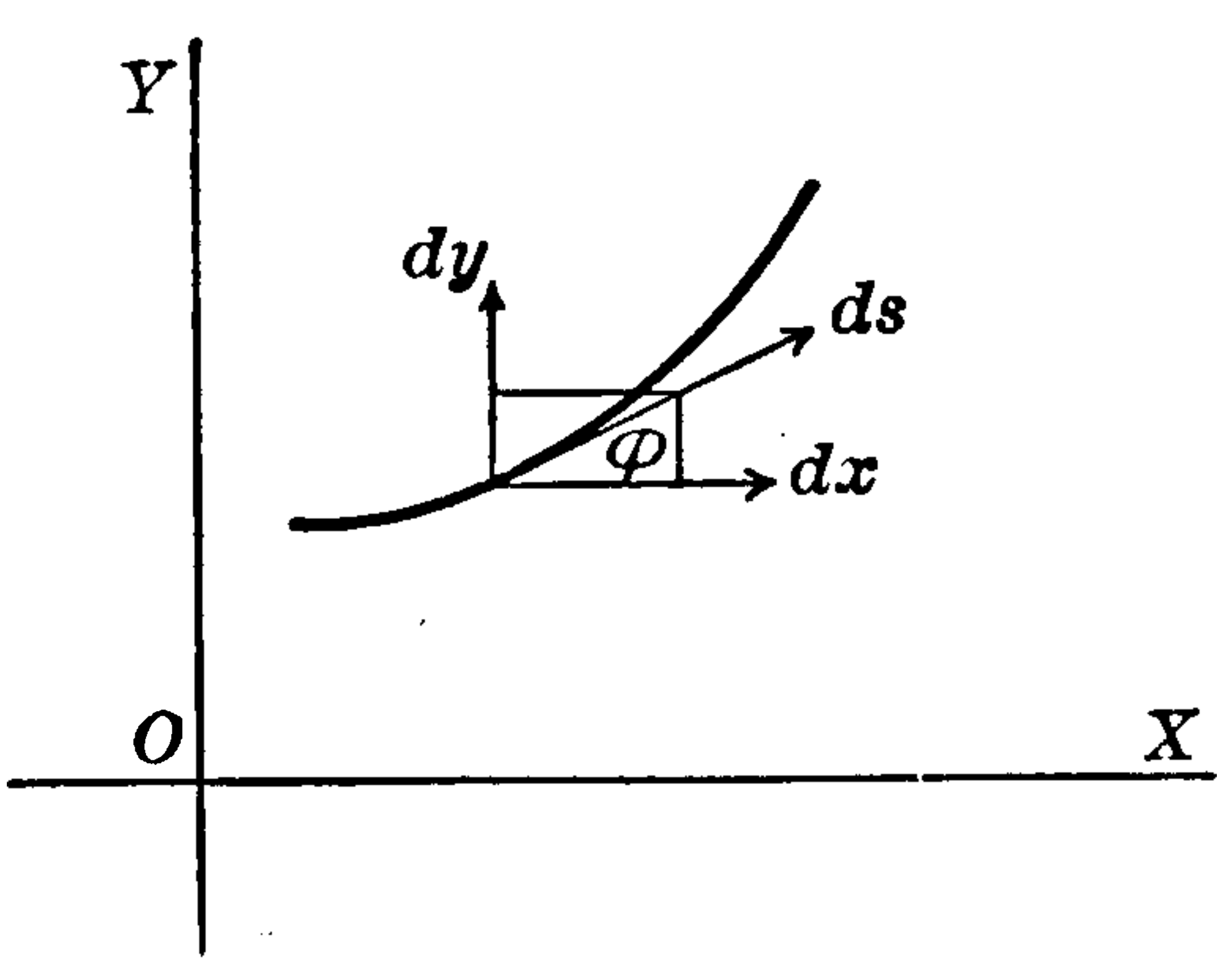


FIG. 30.

85. Derivative of the volume of a solid of revolution. Let the curve APQ revolve about the x-axis, and thus generate a surface of revolution; let V be the volume included between this surface, the plane generated by the fixed ordinate at A, and the plane generated by any ordinate MP.

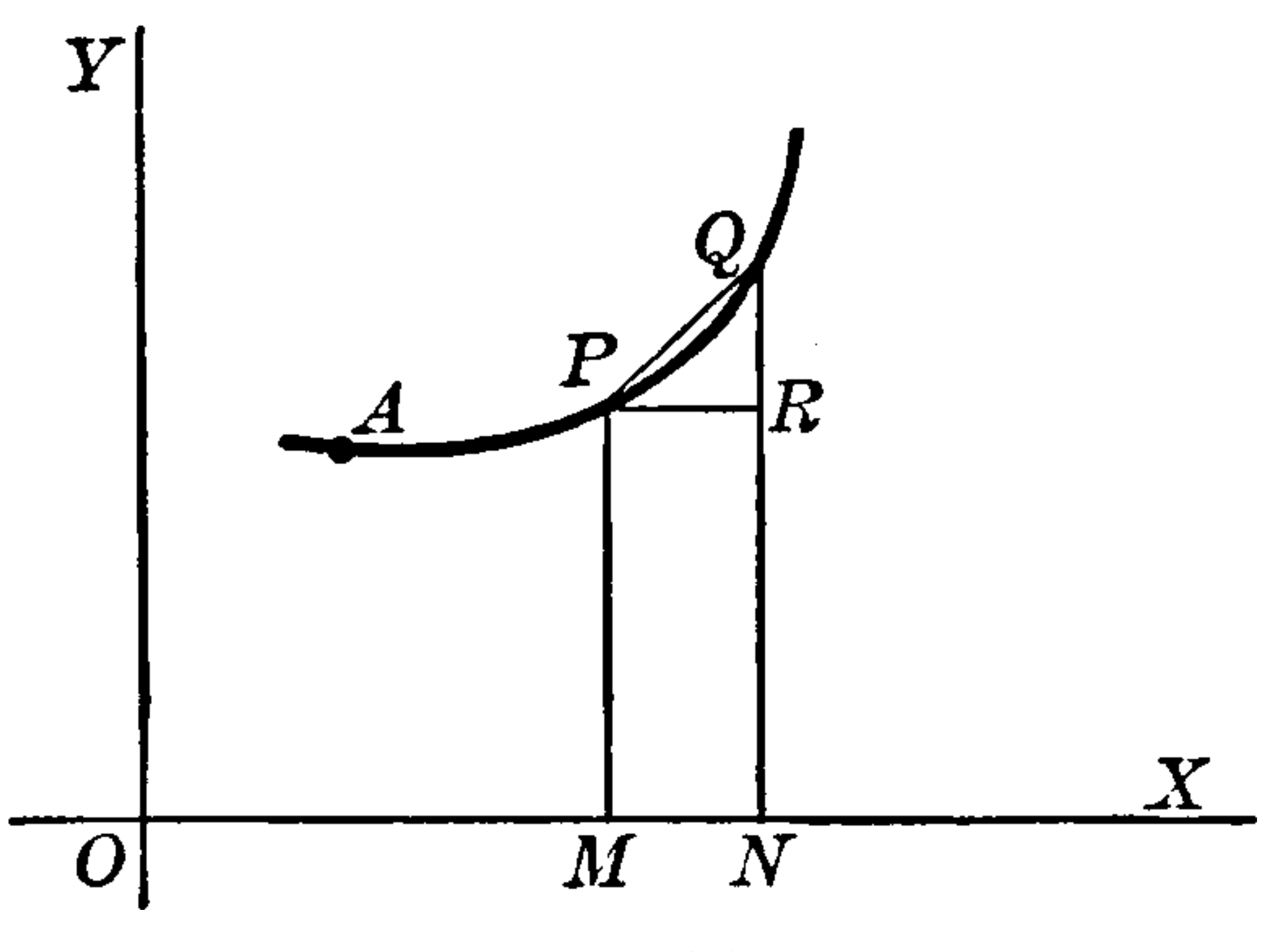


FIG. 30 *a*.

Let ΔV be the volume generated by the area $PMNQ$. Then ΔV lies between the volumes of the cylinders generated by the rectangles $PMNR$ and $SMNQ$; that is,

$$\pi y^2 \Delta x < \Delta V < \pi(y + \Delta y)^2 \Delta x.$$

Dividing by Δx and taking limits,

$$\frac{dV}{dx} = \pi y^2.$$

86. Derivative of a surface of revolution. Let S be the area of the surface generated by the arc AP (Fig. 31), and ΔS that generated by the arc PQ whose length is Δs.

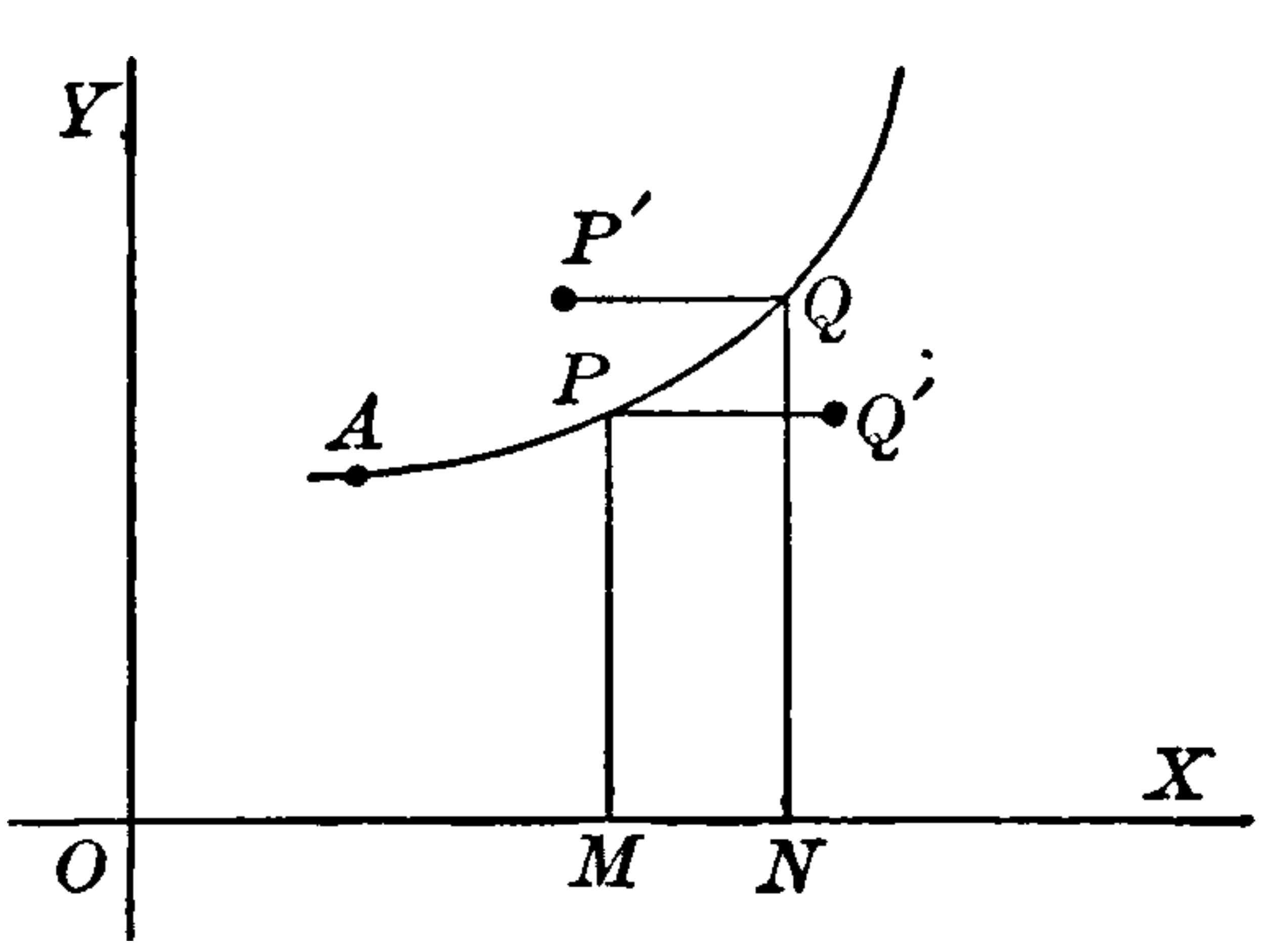


FIG. 31.

Draw PQ', QP' parallel to OX and equal in length to the arc PQ. Then it may be assumed as an axiom that the area generated by PQ lies between the areas generated by PQ' and $P'Q$; *i.e.*,

$$2\pi y \Delta s < \Delta S < 2\pi(y + \Delta y)\Delta s.$$

Dividing by Δs and passing to the limit,

$$\frac{dS}{ds} = 2\pi y, \tag{1}$$

$$\frac{dS}{dx} = \frac{dS}{ds} \cdot \frac{ds}{dx} = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2}$$

87. Derivative of arc in polar coördinates.

Let ρ, θ be the coördinates of P; $\rho+\Delta\rho$, $\theta+\Delta\theta$ those of Q; s the length of the arc KP; Δs that of arc PQ; draw PM perpendicular to OQ. Then

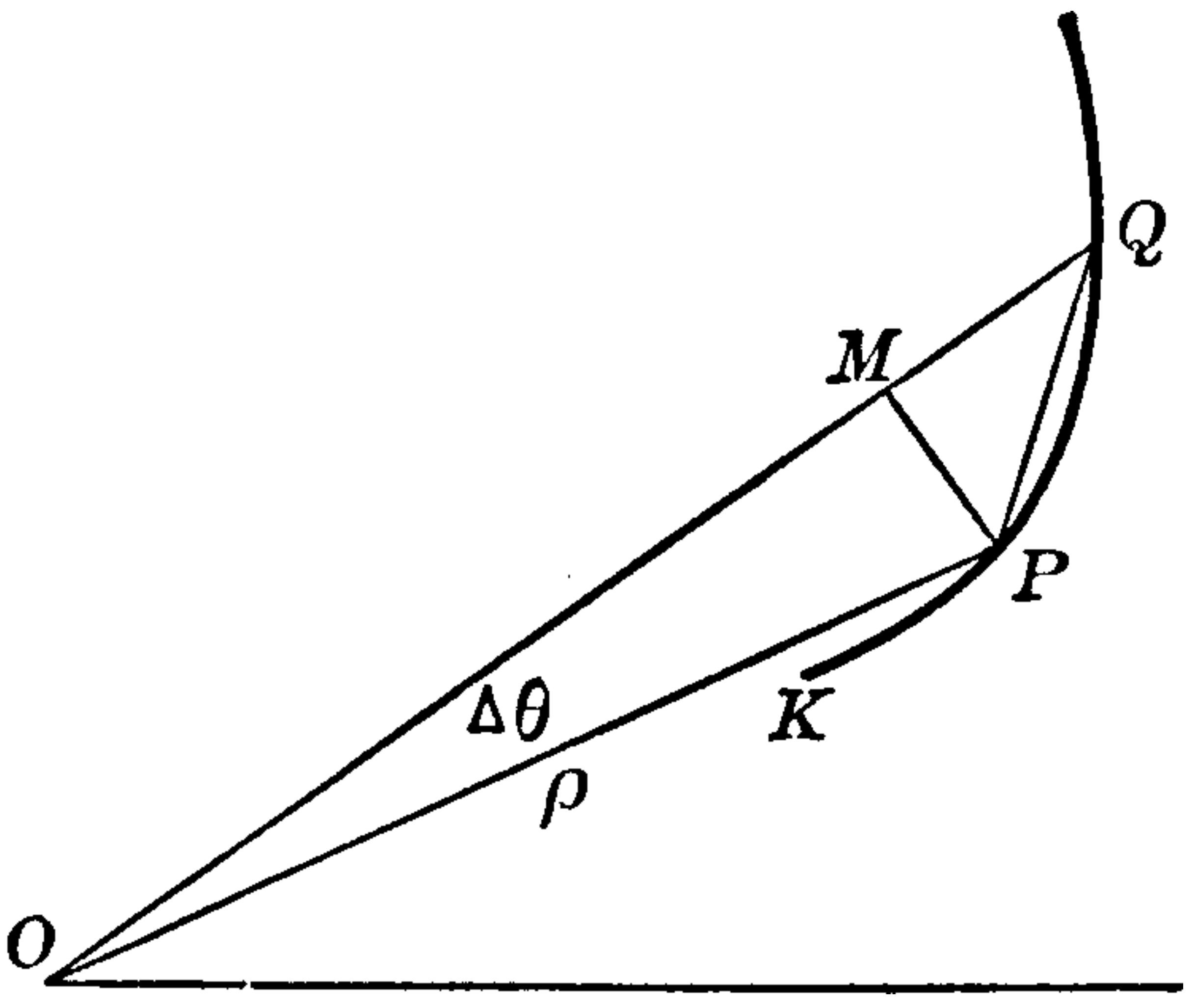


FIG. 32.

$$PM=\rho\sin\Delta\theta,$$

$$MQ=OQ-OM=\rho+\Delta\rho-\rho\cos\Delta\theta$$

$$=\rho(1-\cos\Delta\theta)+\Delta\rho$$

$$=2\rho\sin^2\tfrac{1}{2}\Delta\theta+\Delta\rho.$$

Hence $PQ^2=(\rho\sin\Delta\theta)^2+(2\rho\sin^2\frac{1}{2}\Delta\theta+\Delta\rho)^2$,

$$\left(\frac{PQ}{\Delta\theta}\right)^2=\rho^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2+\left(\rho\sin\tfrac{1}{2}\Delta\theta\cdot\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}+\frac{\Delta\rho}{\Delta\theta}\right)^2.$$

Replacing the first member by $\left(\frac{PQ}{\Delta s}\cdot\frac{\Delta s}{\Delta\theta}\right)^2$, passing to the limit when $\Delta\theta\doteq 0$, and putting $\lim\frac{PQ}{\Delta s}=1$, $\lim\frac{\sin\Delta\theta}{\Delta\theta}=1$, $\lim\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}=1$, it follows that

$$\left(\frac{ds}{d\theta}\right)^2=\rho^2+\left(\frac{d\rho}{d\theta}\right)^2;$$

i.e.,

$$\frac{ds}{d\theta}=\sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2}.$$

In the rate or differential notation this formula may be conveniently written

$$ds^2=d\rho^2+\rho^2d\theta^2.$$

This relation may be readily deduced also from Fig. 26, Art. 80.

88. Derivative of area in polar coördinates. Let A be the area of OKP measured from a fixed radius vector OK to any other radius vector OP; let ΔA be the area of OPQ. Draw arcs PM, QN, with O as a center. Then the area POQ lies between the areas of the sectors OPM and ONQ; *i.e.*,

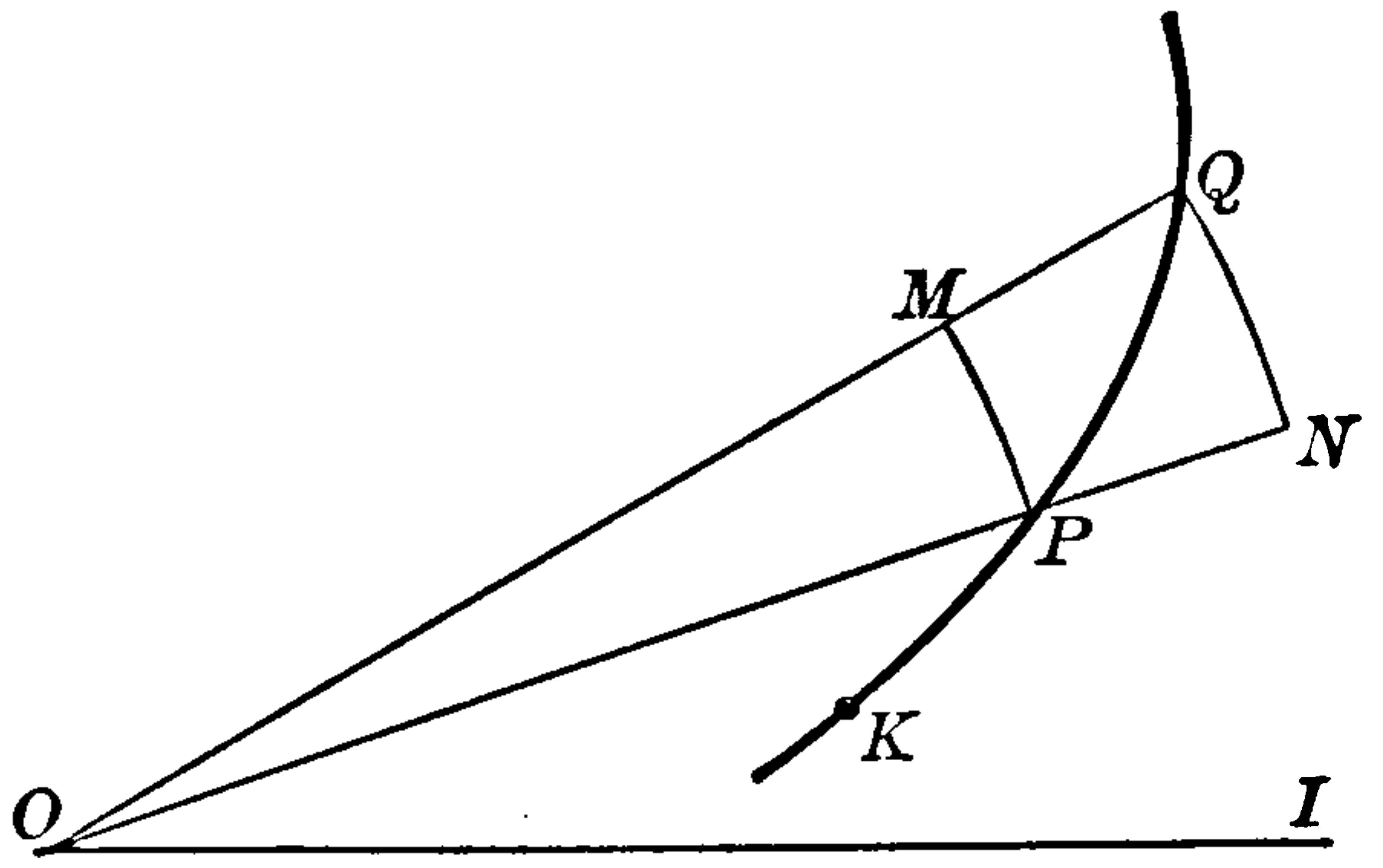


Fig. 33.

$$\tfrac{1}{2}\rho^2\Delta\theta < \Delta A < \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta.$$

Dividing by $\Delta\theta$ and passing to the limit, when $\Delta\theta \doteq 0$, it follows that

$$\frac{dA}{d\theta} = \tfrac{1}{2}\rho^2.$$

For the derivative of the area of a curve in rectangular coördinates, see Art. 10. The result is $\dfrac{dA}{dx} = y.$

EXERCISES ON CHAPTER XI

1. In the parabola $y^2 = 4ax$, find $\dfrac{ds}{dx}, \dfrac{dA}{dx}, \dfrac{dS}{dx}, \dfrac{dV}{dx}.$

2. Find $\dfrac{ds}{dx}$ and $\dfrac{ds}{dy}$ for the circle $x^2 + y^2 = a^2$.

3. Find $\dfrac{ds}{dx}$ for the curve $e^y \cos x = 1$.

4. Find the x-derivative of the volume of the cone generated by revolving the line $y = ax$ about the axis of x.

5. Find the x-derivative of the volume of the ellipsoid of revolution, formed by revolving $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about its major axis.

6. In the curve $\rho = a^\theta$ find $\dfrac{ds}{d\theta}.$ $\dfrac{d\rho}{d\theta} = a^\theta \log a$

7. Given $\rho = a(1 + \cos\theta)$; find $\dfrac{ds}{d\theta}.$

8. In $\rho^2 = a^2 \cos 2\theta$, find $\dfrac{ds}{d\theta}.$

CHAPTER XII

ASYMPTOTES

89. Hyperbolic and parabolic branches. When a curve has a branch extending to infinity, the tangents drawn at successive points of this branch may tend to coincide with a definite fixed line as in the familiar case of the hyperbola. On the other hand, the successive tangents may move farther and farther out of the field as in the case of the parabola. These two kinds of infinite branches may be called *hyperbolic* and *parabolic.*

The character of each of the infinite branches of a curve can always be determined when the equation of the curve is known.

90. Definition of a rectilinear asymptote. If the tangents at successive points of a curve approach a fixed straight line as a limiting position when the point of contact moves farther and farther along any infinite branch of the given curve, then the fixed line is called an *asymptote* of the curve.

This definition may be stated more briefly but less precisely as follows: An asymptote to a curve is a tangent whose point of contact is at infinity, but which is not itself entirely at infinity.

DETERMINATION OF ASYMPTOTES

91. Method of limiting intercepts. The equation of the tangent at any point (x_1, y_1) being

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1),$$

the intercepts made by this line on the coördinate axes are

$$\left.\begin{aligned} y_0 &= y_1 - x_1\frac{dy_1}{dx_1}, \\ x_0 &= x_1 - y_1\frac{dx_1}{dy_1}. \end{aligned}\right\} \qquad (1)$$

Suppose the curve has a branch on which $x \doteq \infty$ and $y \doteq \infty$. Then from (1) the limits can be found to which the intercepts x_0, y_0 approach as the coördinates x_1, y_1 of the point of contact tend to become infinite. If these limits be denoted by a, b, the equation of the corresponding asymptote is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Except in special cases this method is usually too complicated to be of practical use in determining the equations of the asymptotes of a given curve. There are three other principal methods, which will always suffice to determine the asymptotes of curves whose equations involve only algebraic functions. These may be called the methods of inspection, of substitution, and of expansion.

92. Method of inspection. Infinite ordinates, asymptotes parallel to axes. When an algebraic equation in two coördinates x and y is rationalized, cleared of fractions, and arranged according to powers of one of the coördinates, say y, it takes the form

$$ay^n + (bx + c)y^{n-1} + (dx^2 + ex + f)y^{n-2} + \cdots + u_{n-1}y + u_n = 0,$$

in which u_n is a polynomial of the degree n in terms of the other coördinate x, and u_{n-1} is of degree $n - 1$.

When any value is given to x, the equation determines n values for y.

Let it be required to find for what value of x the corresponding ordinate y has an infinite value.

For this purpose the following theorem from algebra will be recalled:

Given an algebraic equation of degree n

$$\alpha y^n + \beta y^{n-1} + \gamma y^{n-2} + \cdots = 0.$$

If $\alpha = 0$, one root y becomes infinite; if $\alpha = 0$ and $\beta = 0$, two roots y become infinite; and in general if the coefficients of each of the k highest powers of y vanish, the equation will have k infinite roots.

Suppose at first that the term in y^n is present; in other words, that the coefficient a is not zero. Then, when any finite value is given to x, all of the n values of y are finite, and there are accordingly no infinite ordinates for finite values of the abscissa.

Next suppose that a is zero, and b, c, not zero. In this case one value of y is infinite for every finite value of x, and hence the curve passes through the point at infinity on the y axis.

There is one particular value of x, namely, $x = \frac{-c}{b}$, for which an additional root of the equation in y becomes infinite. For, when x has this value, the coefficient $bx + c$ of the highest power of y remaining in the equation vanishes.

Geometrically, every line parallel to the y axis has one point of intersection with the curve at infinity, but the line $bx + c = 0$ has two points of intersection with the curve at infinity. A line having two coincident points of intersection with a curve is a tangent to the curve, and when the coincident points are at infinity, but the line itself not altogether at infinity, the tangent is an asymptote. Hence an ordinate that becomes infinite for a definite value of x is an asymptote.

Again, if not only a, but also b and c are zero, there are two values of x that make y infinite; namely, those values of x that make $dx^2+ex+f=0$, and the equations of the infinite ordinates are found by factoring this last equation; and so on.

Similarly, by arranging the equation of the curve according to powers of x, it is easy to find what values of y give an infinite value to x.

Ex. **1.** In the curve

$$2x^3 + x^2y + xy^2 = x^2 - y^2 - 5,$$

find the equation of the infinite ordinate, and determine the finite point in which this line meets the curve.

This is a cubic equation in which the coefficient of y^3 is zero. Arranged in powers of y it is

$$y^2(x+1) + yx^2 + (2x^3 - x^2 + 5) = 0.$$

When $x = -1$, the equation for y becomes

$$0 \cdot y^2 + y + 2 = 0,$$

the two roots of which are $y = \infty$, $y = -2$; hence the equation of the infinite ordinate is $x + 1 = 0$. The infinite ordinate meets the curve again in the finite point $(-1, -2)$.

Since the term in x^3 is present, there are no infinite values of x for finite values of y.

Ex. **2.** Show that the lines $x = a$, and $y = 0$ are asymptotes to the curve $a^2x = y(x-a)^2$ (Fig. 34).

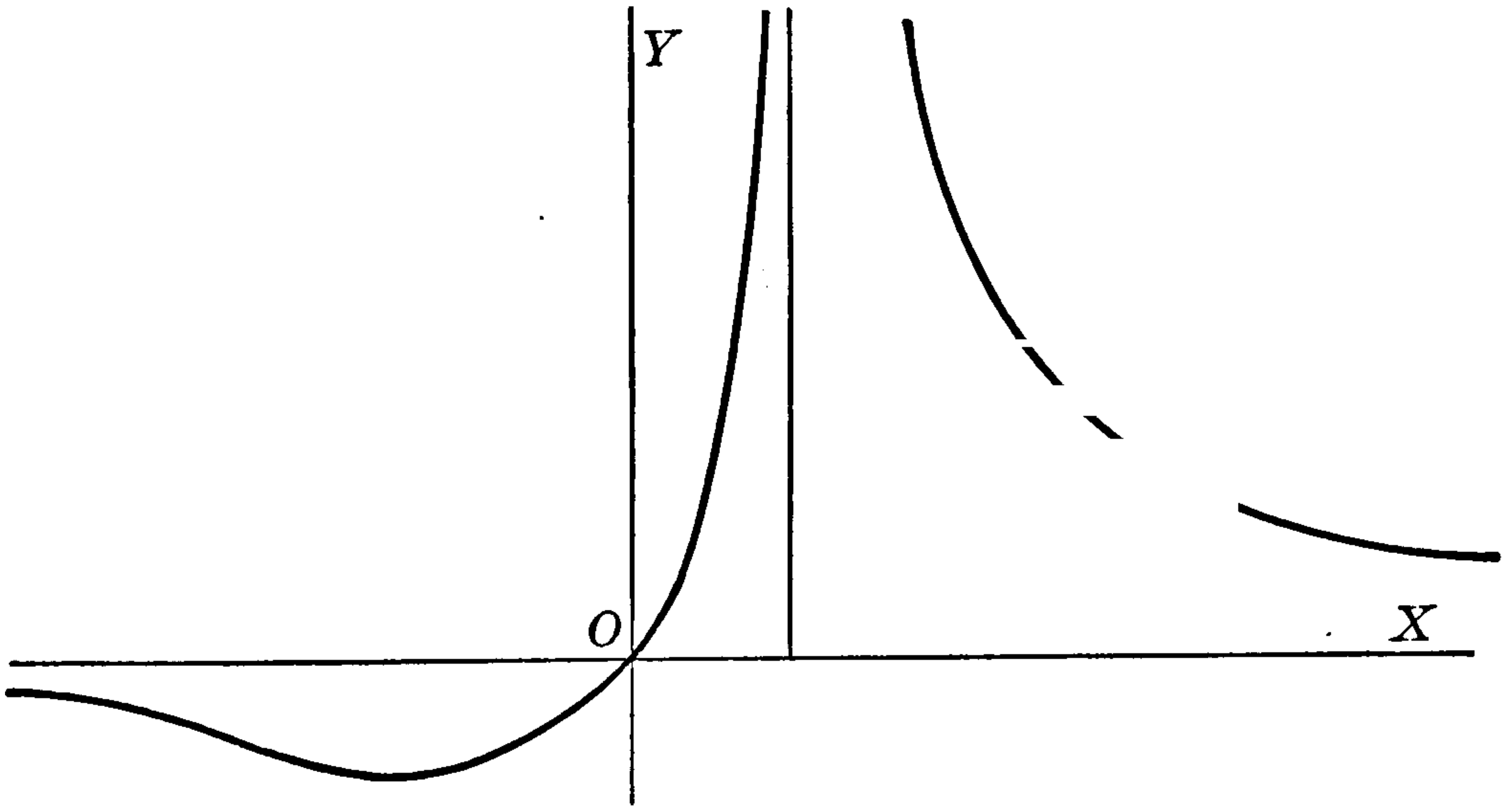


FIG. 34.

Ex. **3.** Find the asymptotes of the curve $x^2(y-a) + xy^2 = a^3$.

93 Method of substitution. Oblique asymptotes. The asymptotes that are not parallel to either axis can be found by the method of substitution, which is applicable to all algebraic curves, and is of especial value when the equation is given in the implicit form

$$f(x, y) = 0. \tag{1}$$

Consider the straight line

$$y = mx + b, \tag{2}$$

and let it be required to determine m and b so that this line shall be an asymptote to the curve $f(x, y) = 0$.

Since an asymptote is the limiting position of a line that meets the curve in two points that tend to coincide at infinity, then, by making (1) and (2) simultaneous, the resulting equation in x,

$$f(x, mx + b) = 0,$$

is to have two of its roots infinite. This requires that the coefficients of the two highest powers of x shall vanish. These coefficients, equated to zero, furnish two equations, from which the required values of m and b can be determined. These values, substituted in (2), will give the equation of an asymptote.

Ex. 4. Find the asymptotes to the curve $y^3 = x^2(2a - x)$.

In the first place, there are evidently no asymptotes parallel to either of the coördinate axes. To determine the oblique asymptotes, make the equation of the curve simultaneous with $y = mx + b$, and eliminate y. Then

$$(mx + b)^3 = x^2(2a - x),$$

or, arranged in powers of x,

$$(1 + m^3)x^3 + (3m^2b - 2a)x^2 + 3b^2mx + b^3 = 0.$$

Let $$m^3 + 1 = 0 \quad \text{and} \quad 3m^2b - 2a = 0.$$

Then $$m = -1,\quad b = \frac{2a}{3};$$

hence $$y = -x + \frac{2a}{3}$$

is the equation of an asymptote.

The third intersection of this line with the given cubic is found from the equation $3mb^2x + b^3 = 0$, whence $x = \frac{2a}{9}$.

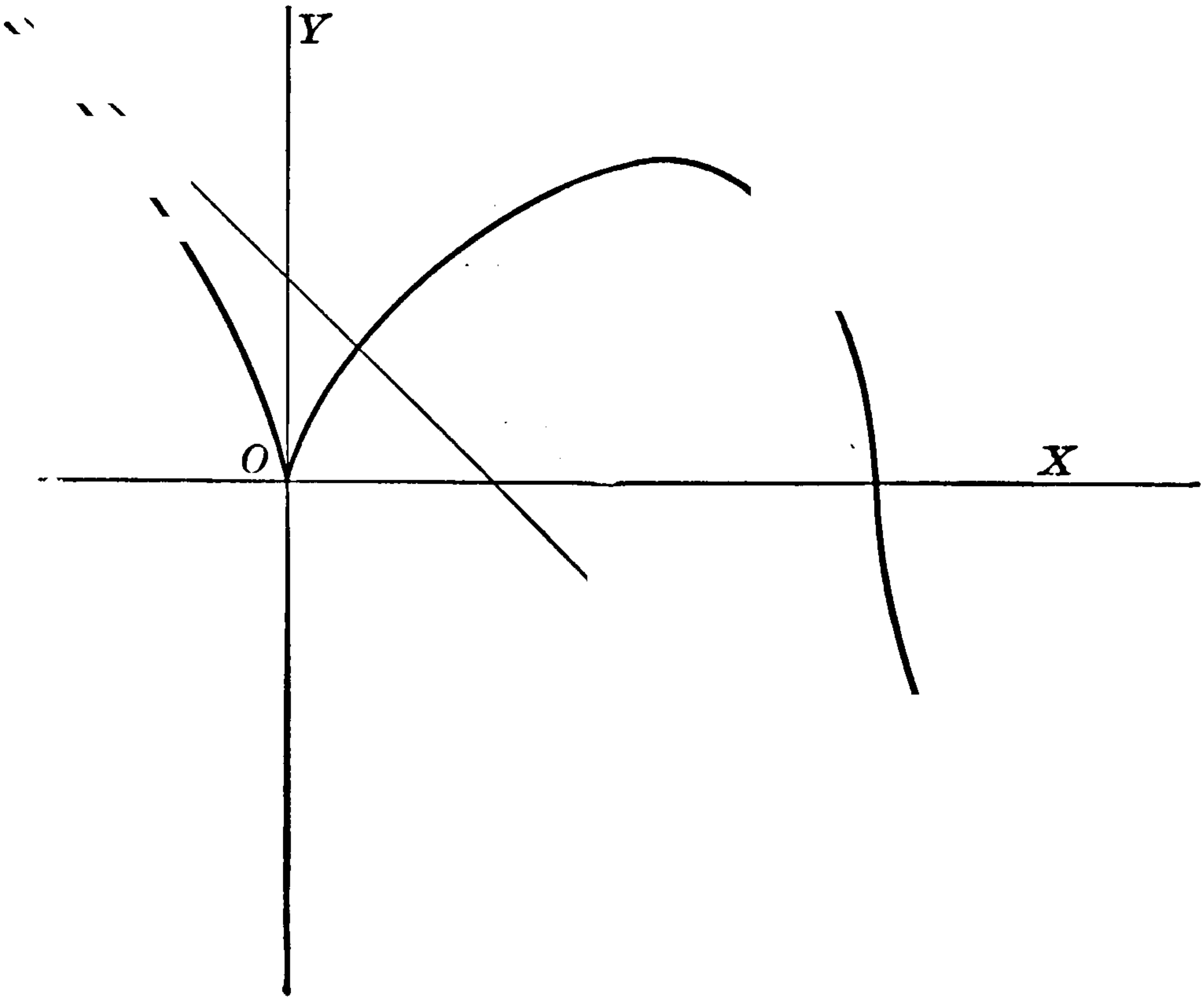


FIG. 35.

This is the only oblique asymptote, as the other roots of the equation for m are imaginary.

Ex. 5. Find the asymptotes to the curve $y(a^2 + x^2) = a^2(a - x)$.

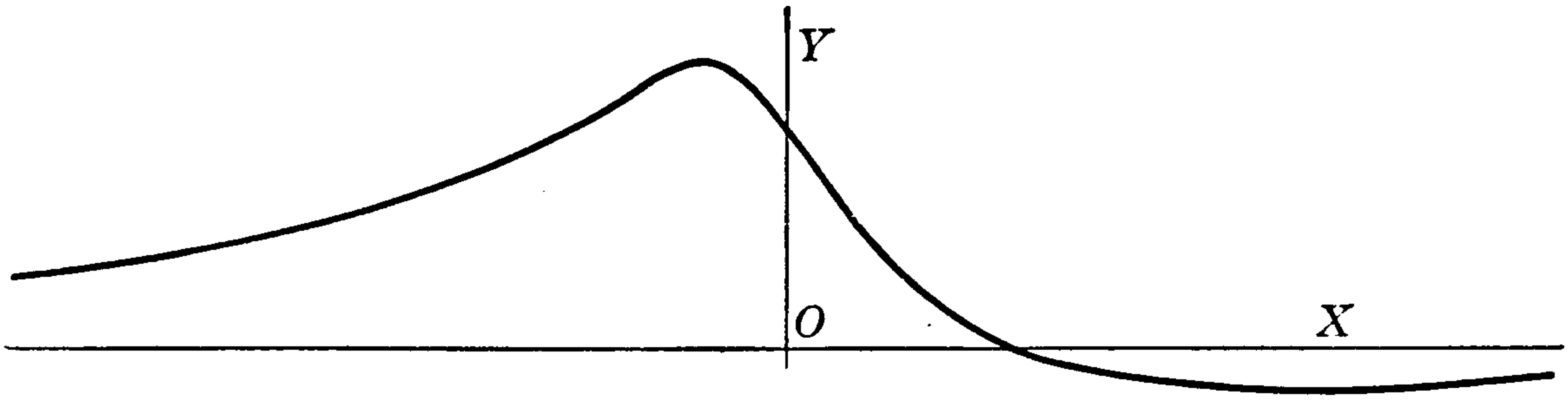


FIG. 36.

Here the line $y = 0$ is a horizontal asymptote by Art. 92. To find the oblique asymptotes, put $y = mx + b$.

Then $$(mx + b)(a^2 + x^2) = a^2(a - x),$$

i.e., $$mx^3 + bx^2 + (ma^2 + a^2)x + (a^2b - a^3) = 0;$$

hence $$m = 0, \quad b = 0, \quad \text{for an asymptote.}$$

Thus the only asymptote is the line $y = 0$ already found.

94. Number of asymptotes. The illustrations of the last article show that if all the terms be present in the general equation of an nth degree curve, then the equation for determining m is of the nth degree and there are accordingly n values of m, real or imaginary. The equation for finding b is usually of the first degree, but for certain curves one or more values of m may cause the coefficient of x^n and x^{n-1} both to vanish, irrespective of b. In such cases any line whose equation is of the form $y = m_1x + c$ will have two points at infinity on the curve independent of c; but by equating the coefficient of x^{n-2} to zero, two values of b can be found such that the resulting lines have three points at infinity in common with the curve. These two lines are parallel; and it will be seen that in each case in which this happens the equation defining m has a double root, so that the total number of asymptotes is not increased. Hence the total number of asymptotes, real and imaginary, is in general equal to the degree of the equation of the curve.

This number must be reduced whenever a curve has a parabolic branch.

Since the imaginary values of m occur in pairs, it is evident that a curve of odd degree has an odd number of real asymptotes; and that a curve of even degree has either no real asymptotes or an even number. Thus, a cubic curve has either one real asymptote or three; a conic has either two real asymptotes or none.

95. Method of expansion. Explicit functions. Although the two foregoing methods are in all cases sufficient to find the asymptotes of algebraic curves, yet in certain special cases the oblique asymptotes are most conveniently found by the method of expansion in descending powers. It is based on the principle that a straight line will be an asymptote to a curve when the difference between the ordinates of the curve and of the line, corresponding to a common abscissa, approaches zero as the abscissa becomes infinite.

It will appear from the process of applying this principle that a line answering the condition just stated will also satisfy the original definition of an asymptote.

The principal value of the method of expansion is that it exhibits the manner in which each infinite branch approaches its asymptote.

Ex. Find the asymptotes of the curve

$$y^2 = \frac{(x-1)(2-x)^2}{x-3}.$$

Here

$$y^2 = \frac{x^2\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)^2}{\left(1-\frac{3}{x}\right)},$$

whence

$$y = \pm x\left(1-\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right)^{-\frac{1}{2}}$$

$$= \pm x\left(1-\frac{1}{2x}-\frac{1}{8x^2}\cdots\right)\left(1-\frac{2}{x}\right)\left(1+\frac{3}{2x}+\frac{27}{8x^2}\cdots\right)$$

$$= \pm x\left(1-\frac{1}{x}+\frac{1}{2x^2}\right) = \pm\left(x-1+\frac{1}{2x}\cdots\right).$$

Hence the oblique asymptotes are $y = \pm(x-1)$ (Fig. 37).

The sign of the next term shows that when $x \doteq +\infty$, the curve is above the first asymptote and below the second; and *vice versa* when $x \doteq -\infty$.

The same method may be applied to cases in which x is an explicit function of y.

This method can also be extended so as to apply to curves defined by an implicit equation, $f(x, y) = 0$. [See McMahon and Snyder's "Differential Calculus," p. 234.]

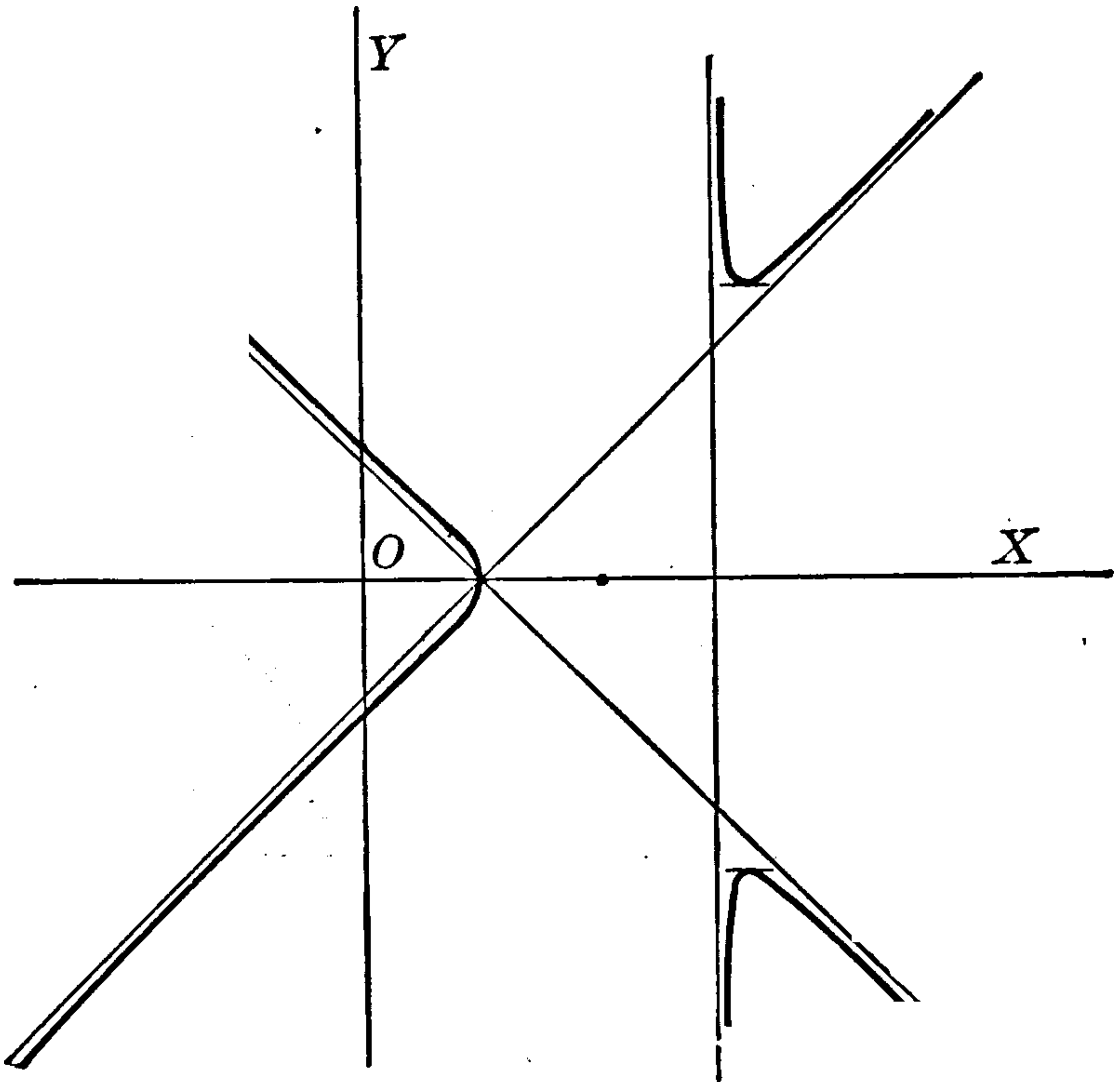


Fig. 37.

EXERCISES ON CHAPTER XII

Find the asymptotes of each of the following curves:

1. $y(a^2 - x^2) = b(2x + c)$.

2. $y^2 = \dfrac{a^2(x-a)(x-3a)}{x^2 - 2ax}$.

3. $x^2y^2 = a^2(x^2 - y^2)$.

4. $y = a + \dfrac{b^3}{(x-c)^2}$.

5. $y^3 = x^2(a - x)$.

6. $y^2(x - 1) = x^2$.

7. $(x + a)y^2 = (y + b)x^2$.

8. $x^2y^2 = x^3 + x + y$.

9. $xy^2 + x^2y = a^3$.

10. $y(x^2 + 3a^2) = x^3$.

11. $x^3 - 3axy + y^3 = 0$.

12. $x^3 + y^3 = a^3$.

13. $x^4 - x^2y^2 + a^2x^2 + b^4 = 0$.

14. $x^4 - y^4 = a^2xy$.

15. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y = 1$.

CHAPTER XIII

DIRECTION OF BENDING. POINTS OF INFLEXION

96. Concavity upward and downward. A curve is said to be *concave downward* in the vicinity of a point P when, for a finite distance on each side of P, the curve is situated below the tangent drawn at that point, as in the arcs AD, FH. It is *concave upward* when the curve lies above the tangent, as in the arcs DF, HK.

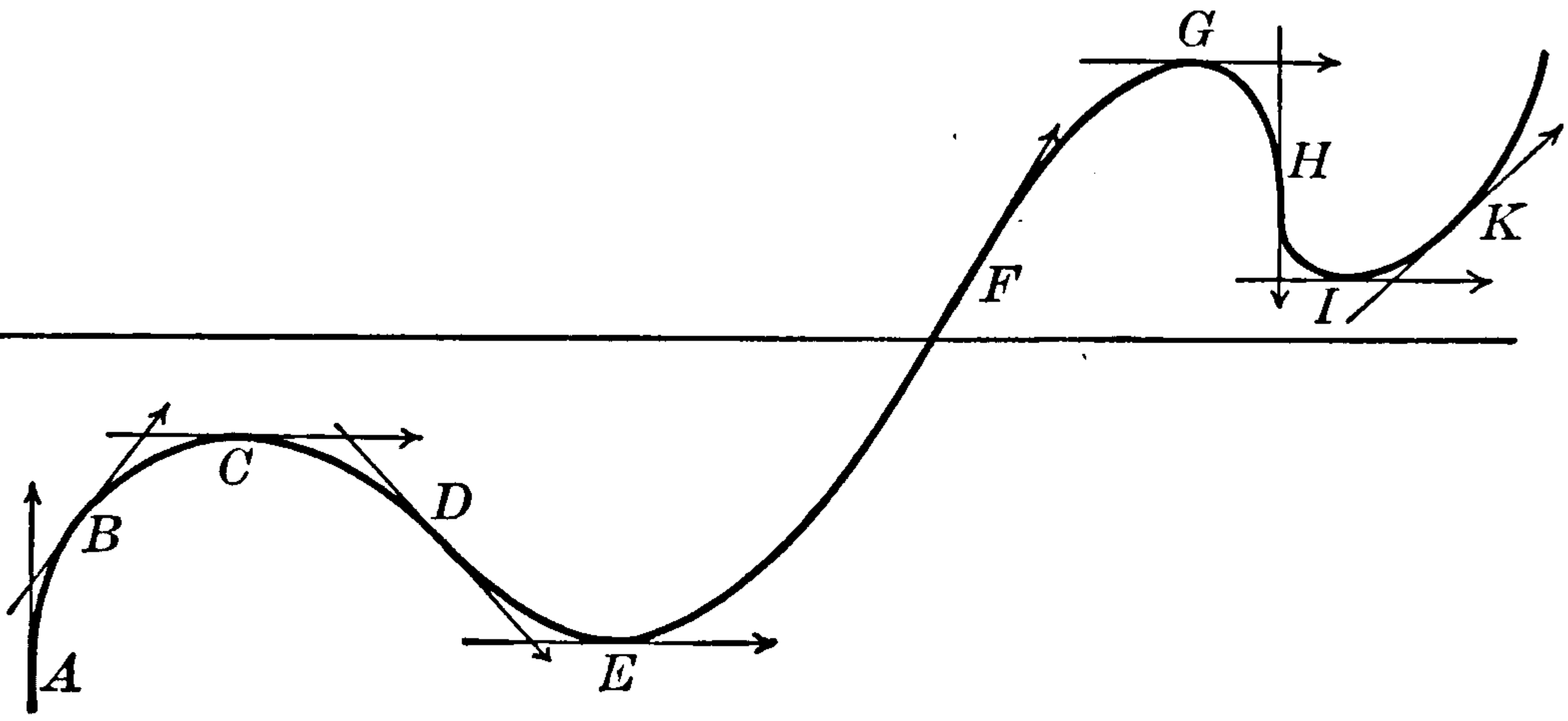


FIG. 38.

By drawing successive tangents to the curve, as in the figure, it is easily seen that if the point of contact advances to the right, the tangent swings in the positive direction of rotation when the concavity is upward, and in the negative direction when the concavity is downward. Hence upward concavity may be called a positive bending of the curve, and downward concavity, negative bending.

A point at which the direction of bending changes continuously from positive to negative, or *vice versa*, as at F or at D, is called a *point of inflexion*, and the tangent at such a point is called a *stationary tangent*.

The points of the curve that are situated just before and just after the point of inflexion are thus on opposite sides of the stationary tangent, and hence the tangent crosses the curve, as at D, F, H.

97. Algebraic test for positive and negative bending. Let the inclination of the tangent line, measured from the right-hand end of the x-axis toward the forward (right-hand) end of the tangent, be denoted by ϕ. Then ϕ is an increasing or decreasing function of the abscissa according as the bending is positive or negative; for instance, in the arc AD, the angle ϕ diminishes from $+\frac{\pi}{2}$ through zero to $-\frac{\pi}{4}$; in the arc DF, ϕ increases from $-\frac{\pi}{4}$ through zero to $\frac{\pi}{3}$; in the arc FH, ϕ decreases from $+\frac{\pi}{3}$ through zero to $-\frac{\pi}{2}$; and in the arc HK, ϕ increases from $-\frac{\pi}{2}$ through zero to $+\frac{\pi}{4}$.

At a point of inflexion ϕ has evidently a turning value which is a maximum or minimum, according as the concavity changes from upward to downward, or conversely.

Thus in Fig. 38, ϕ is a maximum at F, and a minimum at D and at H.

Instead of recording the variation of the angle ϕ, it is generally convenient to consider the variation of the slope $\tan\phi$, which is easily expressed as a function of x by the equation

$$\tan\phi = \frac{dy}{dx}.$$

Since $\tan\phi$ is always an increasing function of ϕ, it follows that the slope function $\frac{dy}{dx}$ is an increasing or a decreasing function of x, according as the concavity is upward or downward, and hence that its x-derivative is positive or negative.

Thus the bending of the curve is in the positive or negative direction of rotation, according as the function $\frac{d^2y}{dx^2}$ is positive or negative.

At a point of inflexion the slope $\frac{dy}{dx}$ is a maximum or minimum, and therefore its derivative $\frac{d^2y}{dx^2}$ changes sign from positive to negative or from negative to positive. This latter condition is evidently both necessary and sufficient in order that the point (x, y) may be a point of inflexion on the given curve.

Hence, the coördinates of the points of inflexion on the curve

$$y = f(x)$$

may be found by solving the equations

$$f''(x) = 0, \quad f''(x) = \infty,$$

and then testing whether $f''(x)$ changes its sign as x passes through the critical values thus obtained. To any critical value a that satisfies the test, corresponds the point of inflexion $(a, f(a))$.

Ex. 1. For the curve

$$y = (x^2 - 1)^2$$

find the points of inflexion, and show the mode of variation of the slope and of the ordinate.

Here

$$\frac{dy}{dx} = 4x(x^2 - 1),$$

$$\frac{d^2y}{dx^2} = 4(3x^2 - 1),$$

hence the critical values for inflexions are $x = \pm\frac{1}{\sqrt{3}}$. It will be seen that as x increases through $-\frac{1}{\sqrt{3}}$, the second derivative changes sign from positive to negative, hence there is an inflexion at which the concavity changes from upward to downward. Similarly, at $x = +\frac{1}{\sqrt{3}}$ the concavity changes from downward to upward. The following numerical table will help to show the mode of variation of the ordinate and of the slope, and the direction of bending.

x	y	$\dfrac{dy}{dx}$	$\dfrac{d^2y}{dx^2}$
$-\infty$	$+\infty$	$-\infty$	$+$
-2	$+25$	-24	$+$
-1	0	0	$+$
$-\dfrac{1}{\sqrt{3}}$	$+\dfrac{4}{9}$	$\dfrac{8}{3\sqrt{3}}$	0
0	1	0	$-$
$+\dfrac{1}{\sqrt{3}}$	$+\dfrac{4}{9}$	$-\dfrac{8}{3\sqrt{3}}$	0
1	0	0	$+$
$+\infty$	$+\infty$	$+\infty$	$+$

As x increases from $-\infty$ to $-\dfrac{1}{\sqrt{3}}$ the bending is positive, and the slope continually increases from $-\infty$ through zero to a maximum value $\dfrac{+8}{3\sqrt{3}}$, which is the slope of the stationary tangent drawn at the point $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{4}{9}\right)$.

As x continues to increase from $-\dfrac{1}{\sqrt{3}}$ to $+\dfrac{1}{\sqrt{3}}$, the bending is negative, and the slope decreases from $+\dfrac{8}{3\sqrt{3}}$ through zero to a minimum value $\dfrac{-8}{3\sqrt{3}}$, which is the slope of the stationary tangent at $\left(+\dfrac{1}{\sqrt{3}}, \dfrac{4}{9}\right)$.

Finally, as x increases from $+\dfrac{1}{\sqrt{3}}$ to $+\infty$, the bending is positive and the slope increases from the value $-\dfrac{8}{3\sqrt{3}}$ through zero to $+\infty$.

The values $x=-1$, 0, $+1$, at which the slope passes through zero, correspond to turning values of the ordinate.

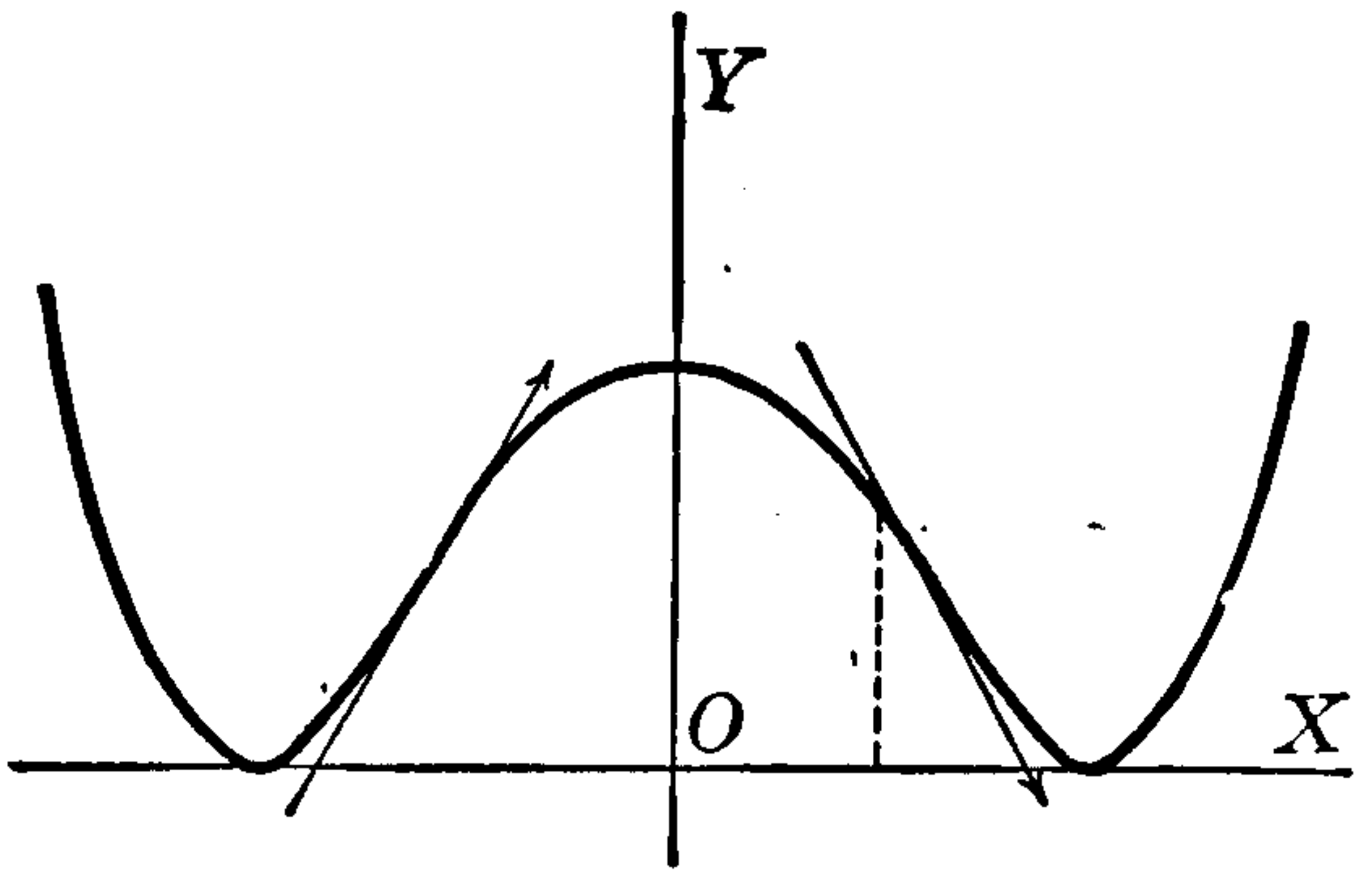


FIG. 39.

Ex. 2. Examine for inflexions the curve $x+4=(y-2)^3$.

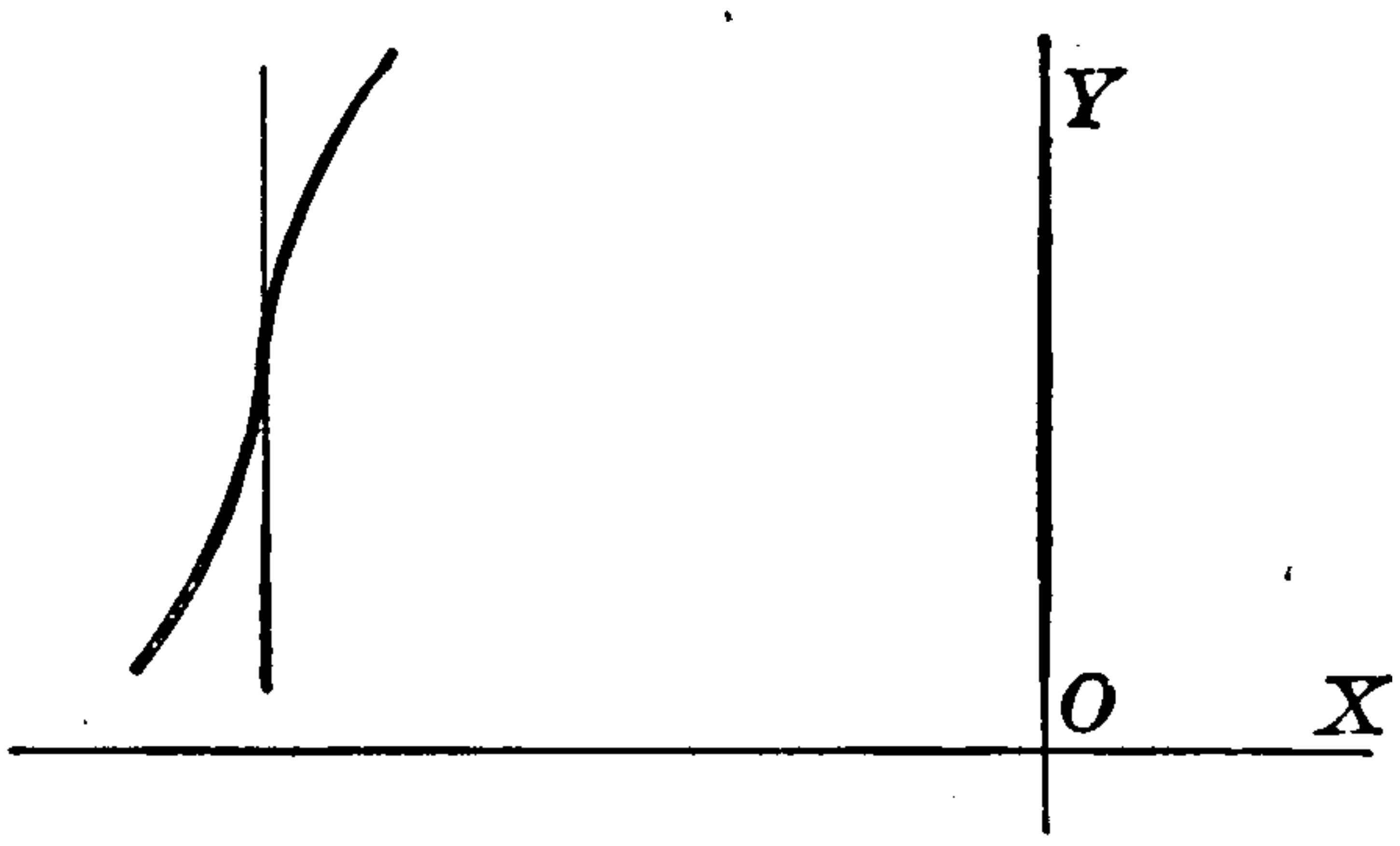


FIG. 40.

In this case

$$y = 2 + (x+4)^{\frac{1}{3}},$$

$$\frac{dy}{dx} = \frac{1}{3}(x+4)^{-\frac{2}{3}},$$

$$\frac{d^2y}{dx^2} = -\frac{2}{9}(x+4)^{-\frac{5}{3}}.$$

Hence, at the point $(-4, 2)$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are infinite. When $x<-4$, $\dfrac{d^2y}{dx^2}$ is positive, and when $x>-4$, $\dfrac{d^2y}{d^2x}$ is negative.

Thus there is a point of inflexion at $(-4, 2)$, at which the slope is infinite, and the bending changes from the positive to the negative direction.

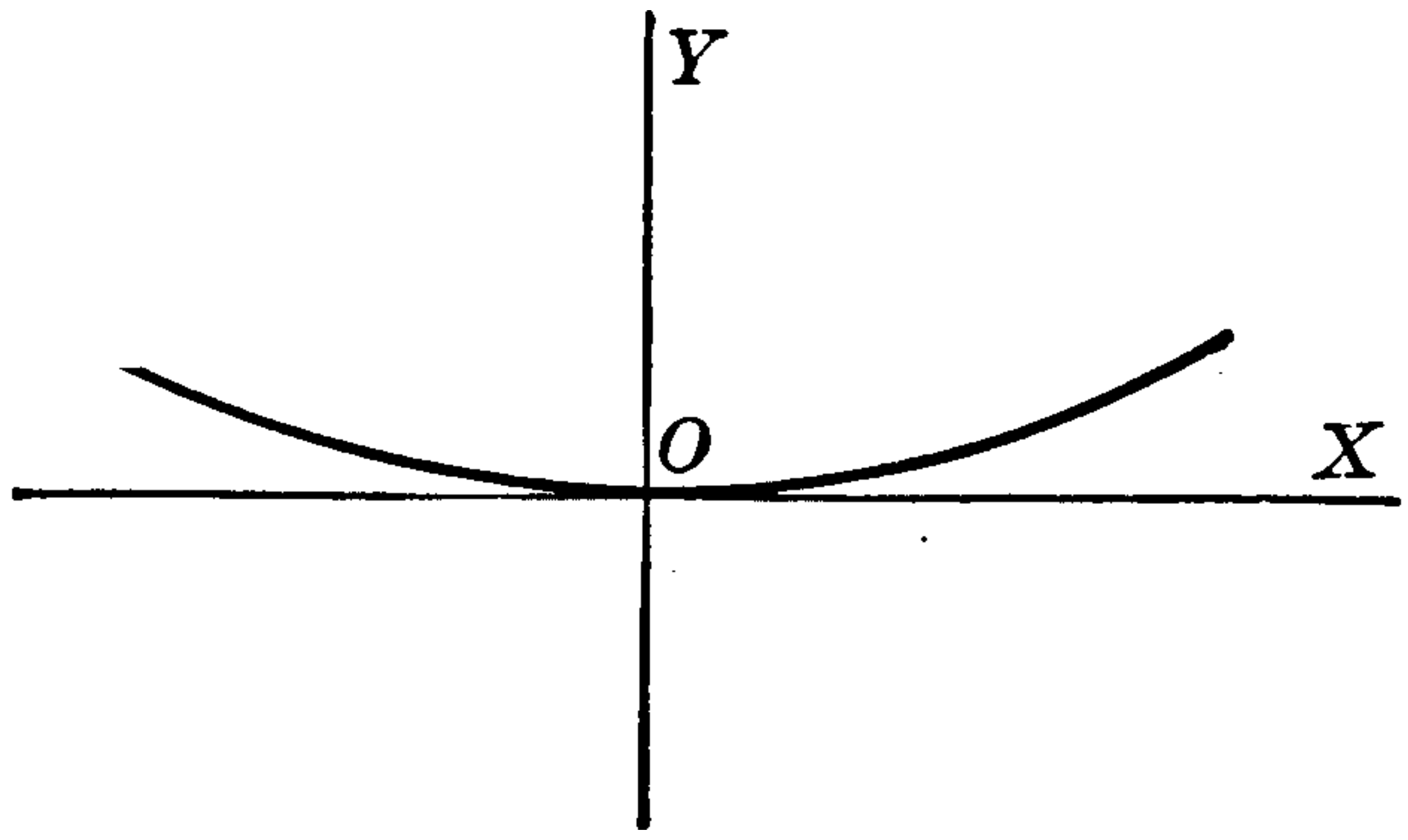


FIG. 41.

Ex. 3. Consider the curve $y=x^4$.

$$\frac{dy}{dx}=4x^3,\ \frac{d^2y}{dx^2}=12x^2.$$

At $(0, 0)$, $\frac{d_2y}{dx^2}$ is zero, but the curve has no inflexion, for $\frac{d^2y}{dx^2}$ never changes sign (Fig. 41).

98. Analytical derivation of the test for the direction of bending. Let the equation of a curve be $y=f(x)$, and let P, (x_1, y_1), be a point upon it. Then the equation of the tangent at P is

$$y-y_1=f'(x_1)(x-x_1).$$

Suppose that when x changes from x_1 to x_1+h, the ordinate of the tangent change from y_1 to y', and that of the curve from y_1 to y''; then it is proposed to determine the sign of the difference of ordinates $y''-y'$ corresponding to the same abscissa x_1+h.

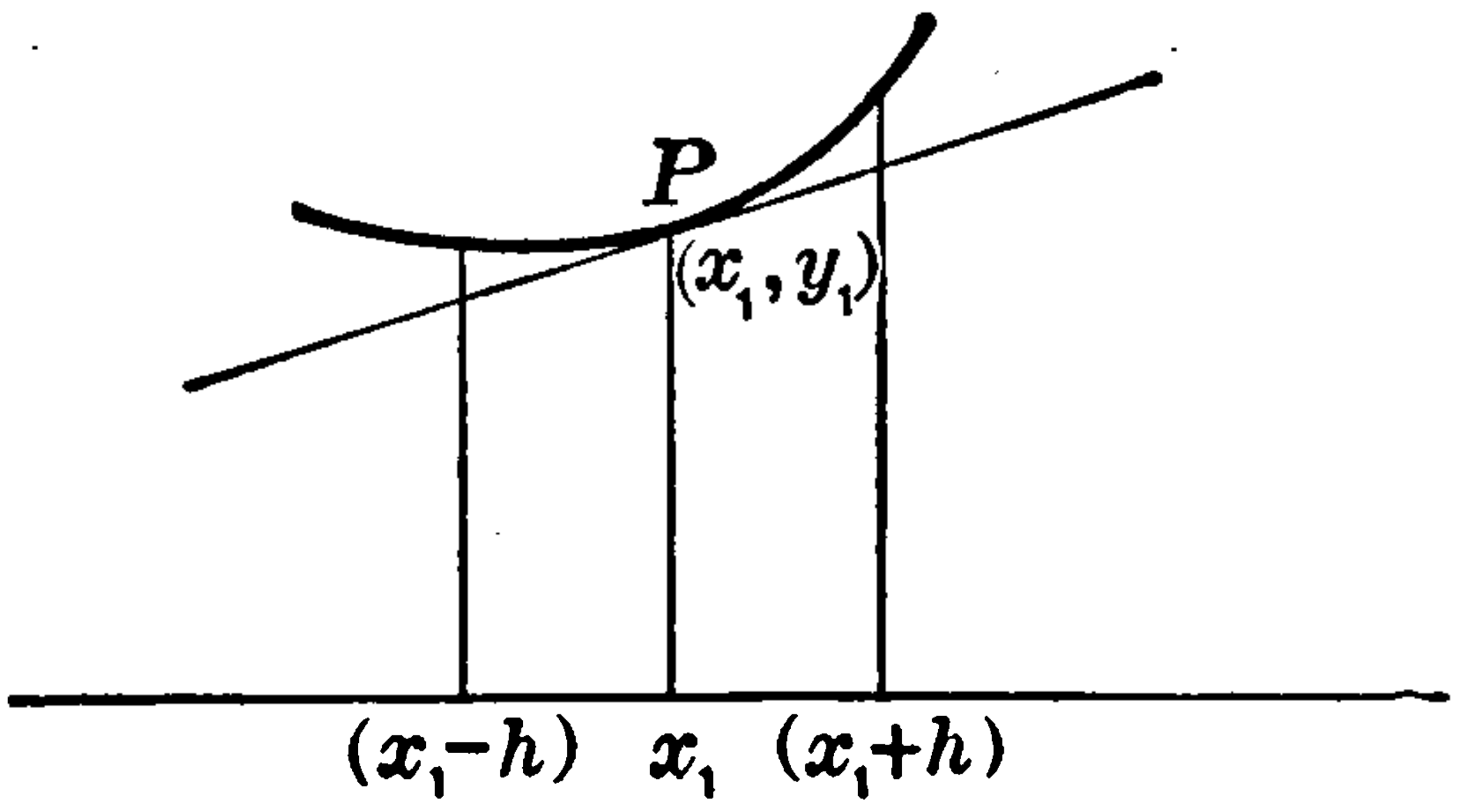


FIG. 42.

By Taylor's theorem,

$$y''=f(x_1+h)=f(x_1)+hf'(x_1)+\frac{h^2}{1\cdot 2}f''(x_1)+\cdots;$$

and from the above equation of the tangent,

$$y'-y_1=f'(x_1)\cdot(x_1+h-x_1).$$

Hence $$y'=y_1+hf'(x_1)=f(x_1)+hf'(x_1),$$

and it follows that

$$y''-y'=\frac{h^2}{2}f''(x_1)+\cdots.$$

When h is made sufficiently small, $f''(x_1) + \cdots$ will have the same sign as $f''(x_1)$; but the factor h^2 is always positive, hence when $f''(x_1)$ is positive, $y'' - y'$ is positive, and thus the curve is above the tangent at both sides of the point of contact, that is, the concavity is upward. Similarly, when $f''(x_1)$ is negative, the concavity is downward.

This agrees with the former result.

99. Concavity and convexity towards the axis. A curve is said to be convex or concave toward a line, in the vicinity of a given point on the curve, according as the tangent at the point does or does not lie between the curve and the line, for a finite distance on each side of the point of contact.

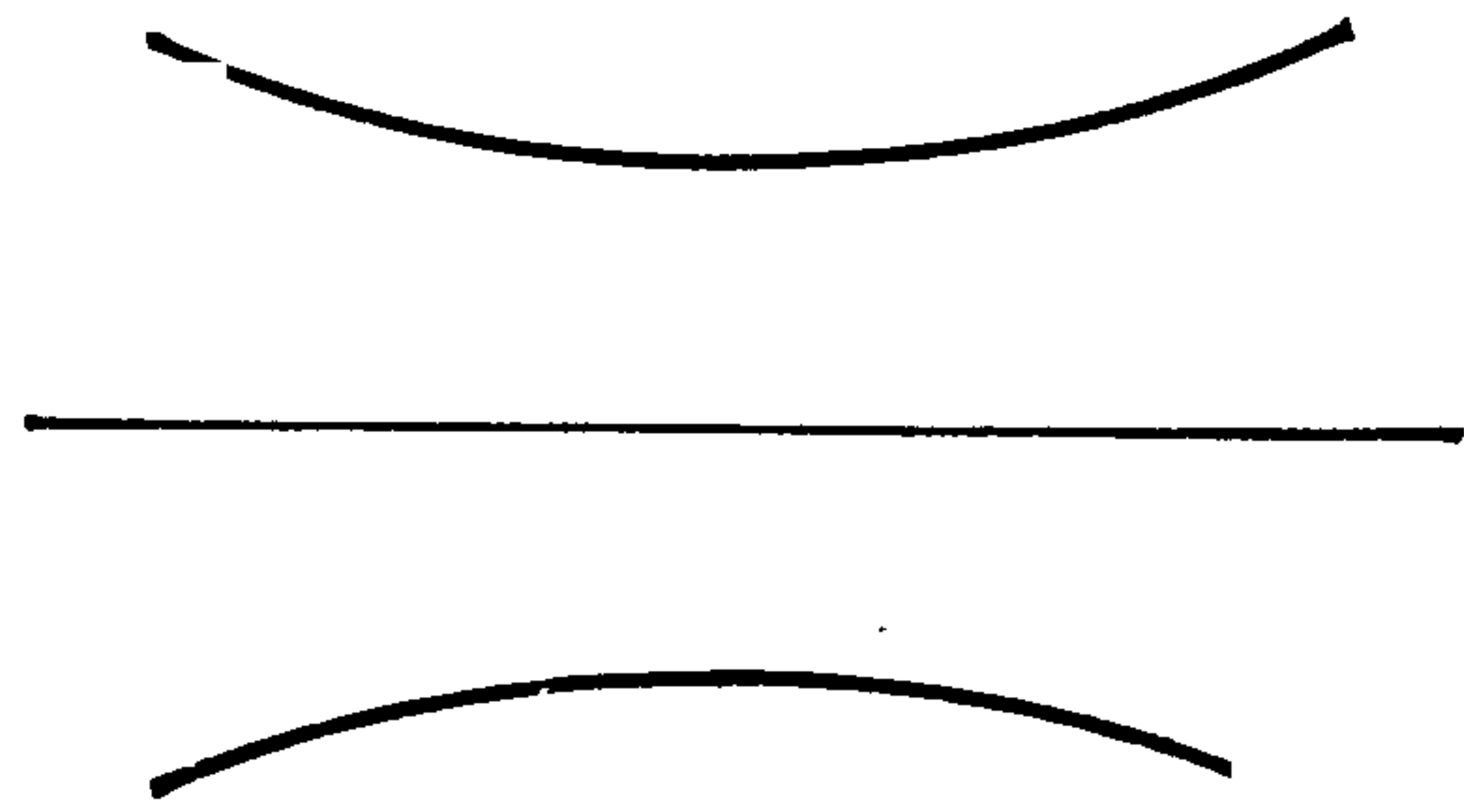


FIG. 43 *a*.

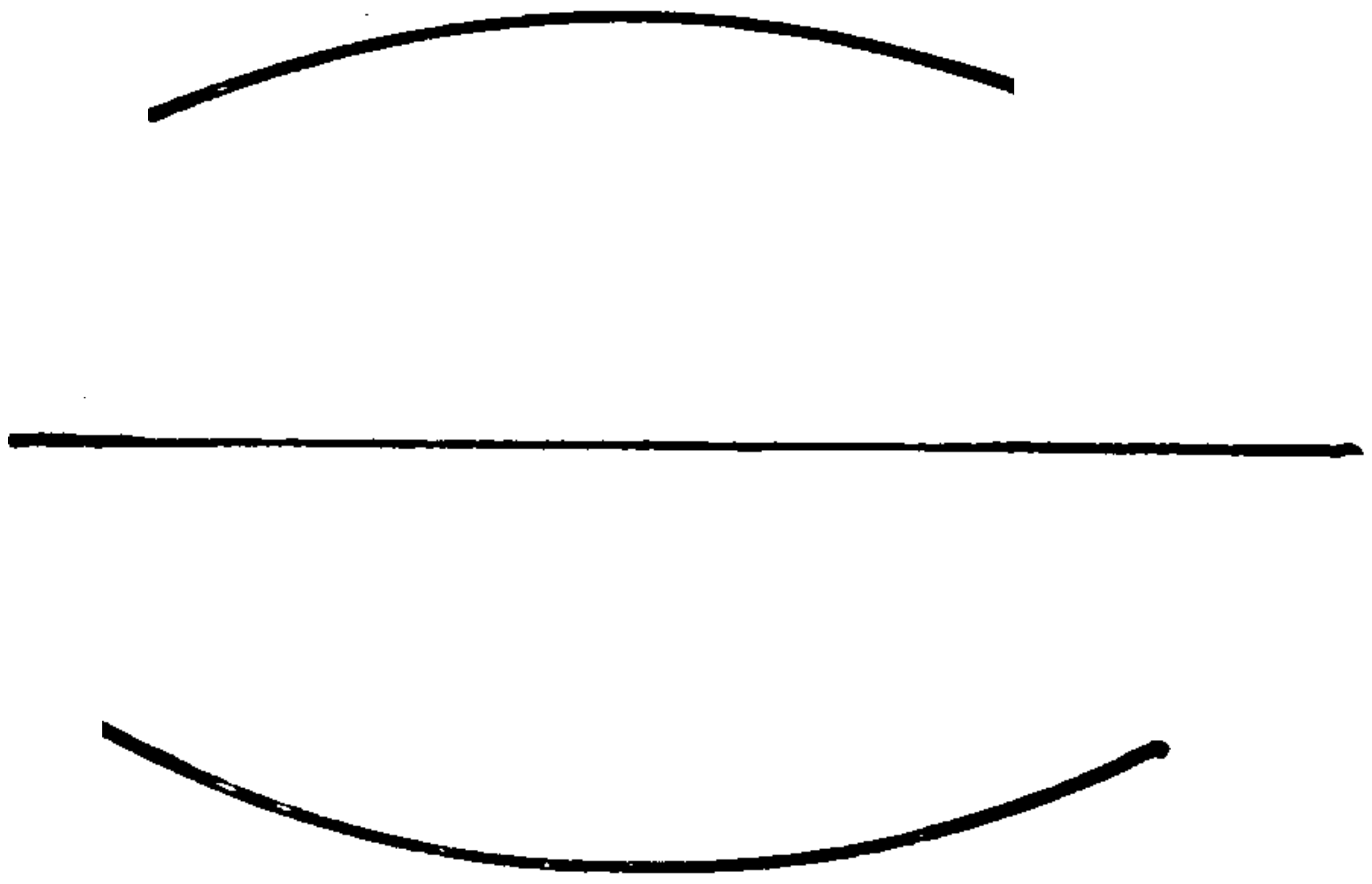


FIG. 43 *b*.

First, let the curve be convex toward the x-axis, as in the left-hand figure. Then if y is positive, the bending is positive and $\frac{d^2y}{dx^2}$ is positive; but if y is negative, the bending is negative and $\frac{d^2y}{dx}$ is negative. Hence in either case the product $y\frac{d^2y}{dx^2}$ is positive.

Next, let the curve be concave toward the x-axis, as in the right-hand figure. Then if y is positive, the bending is negative and $\frac{d^2y}{dx}$ is negative; but if y is negative, the bending is positive and $\frac{d^2y}{dx^2}$ is positive. Thus in either case the product $y\frac{d^2y}{dx^2}$ is negative. Hence:

In the vicinity of a given point (x, y) *the curve is convex or concave to the x-axis, according as the product* $y\dfrac{d^2y}{dx^2}$ *is positive or negative.*

EXERCISES ON CHAPTER XIII

1. Examine the curve $y = 2 - 3(x - 2)^{\frac{3}{5}}$ for points of inflexion.

2. Show that the curve $a^2y = x(a^2 - x^2)$ has a point of inflexion at the origin.

3. Find the points of inflexion on the curve $y = \dfrac{8a^3}{x^2 + 4a^2}$.

4. In the curve $ay = x^{\frac{m}{n}}$, prove that the origin is a point of inflexion if m and n are positive odd integers.

5. Show that the curve $y = c\sin\dfrac{x}{a}$ has an infinite number of points of inflexion lying on a straight line.

6. Show that the curve $y(x^2 + a^2) = x$ has three points of inflexion lying on a straight line; find the equation of the line.

7. If $y^2 = f(x)$ be the equation of a curve, prove that the abscissas of its points of inflexion satisfy the equation

$$[f'(x)]^2 = 2f(x)\cdot f''(x).$$

8. Draw the part of the curve $a^2y = \dfrac{x^3}{3} - ax^2 + 2a^3$ near its point of inflexion, and find the equation of the stationary tangent.

CHAPTER XIV

CONTACT AND CURVATURE

100. Order of contact. The points of intersection of the two curves

$$y = \phi(x), \quad y = \psi(x)$$

are found by making the two equations simultaneous; that is, by finding those values of x for which

$$\phi(x) = \psi(x).$$

Suppose $x = a$ is one value that satisfies this equation. Then the point $x = a$, $y = \phi(a) = \psi(a)$ is common to the curves.

If, moreover, the two curves have the same tangent at this point, they are said to touch each other, or to have *contact* of the first order with each other. The values of y and of $\frac{dy}{dx}$ are thus the same for both curves at the point in question, which requires that

$$\phi(a) = \psi(a),$$
$$\phi'(a) = \psi'(a).$$

If, in addition, the value of $\frac{d^2y}{dx^2}$ be the same for each curve at the point, then

$$\phi''(a) = \psi''(a),$$

and the curves are said to have a contact of the second order with each other.

If $\phi(a) = \psi(a)$, and all the derivatives up to the nth order inclusive be equal to each other, the curves are said to have contact of the nth order. This is seen to require $n+1$ conditions. Hence if the equation of the curve $y=\phi(x)$ be given, and if the equation of a second curve be written in the form $y=\psi(x)$, in which $\psi(x)$ proceeds in powers of x with undetermined coefficients, then $n+1$ of these coefficients could be determined by requiring the second curve to have contact of the nth order with the given curve at a given point.

101. Number of conditions implied by contact. A straight line has two arbitrary constants, which can be determined by two conditions; accordingly a straight line can be drawn which touches a given curve at any specified point. For if the equation of a line be written $y = mx + b$, then

$$\frac{dy}{dx} = m, \quad \frac{d^2y}{dx^2} = 0;$$

hence, through any arbitrary point $x = a$ on a given curve $y = \phi(x)$, a line can be drawn which has contact of the first order with the curve, but which has not in general contact of the second order; for the two conditions for first-order contact are

$$\begin{aligned} ma + b &= \phi(a), \\ m \quad &= \phi'(a), \end{aligned}$$

which are just sufficient to determine m and b.

In general no line can be drawn having contact of an order higher than the first with a given curve; but there are certain points at which this can be done. For example, the additional condition for second-order contact is $0 = \phi''(a)$, which is satisfied when the point $x = a$ is a point of inflexion on the given curve $y = \phi(x)$. Thus the tangent at a point of inflexion on a curve has contact of the second order with the curve.

The equation of a circle has three independent constants. It is therefore possible to determine a circle having contact of the second order with a given curve at any assigned point.

The equation of a parabola has four constants, hence a parabola can be found which has contact of the third order with the given curve at any point.

The general equation of a central conic has five independent constants, hence a conic can be found which has contact of the fourth order with a given curve at any specified point.

As in the case of the tangent line, special points may be found for which these curves have contact of higher order.

102. Contact of odd and of even order.

THEOREM. At a point where two curves have contact of an odd order they do not cross each other; but they do cross where they have contact of an even order.

For, let the curves $y=\phi(x)$, $y=\psi(x)$ have contact of the nth order at the point whose abscissa is a; and let y_1, y_2 be the ordinates of these curves at the point whose abscissa is $a+h$. Then

$$y_1=\phi(a+h), \quad y_2=\psi(a+h),$$

and by Taylor's theorem

$$y_1=\phi(a)+\phi'(a)\cdot h+\frac{\phi''(a)}{2!}\cdot h^2+\cdots$$
$$+\frac{\phi^n(a)}{n!}\cdot h^n+\frac{h^{n+1}}{(n+1)!}\phi^{n+1}(a)+\cdots;$$

$$y_2=\psi(a)+\psi'(a)\cdot h+\frac{\psi''(a)}{2!}\cdot h^2+\cdots$$
$$+\frac{\psi^n(a)}{n!}\cdot h^n+\frac{h^{n+1}}{(n+1)!}\cdot\psi^{n+1}(a)+\cdots.$$

Since by hypothesis the two curves have contact of the nth order at the point whose abscissa is a, hence

$$\phi(a) = \psi(a), \quad \phi'(a) = \psi'(a), \quad \cdots, \quad \phi^n(a) = \psi^n(a),$$

and $$y_1 - y_2 = \frac{h^{n+1}}{(n+1)!}[\phi^{n+1}(a) + \cdots - \psi^{n+1}(a) - \cdots];$$

but this expression, when h is sufficiently diminished, has the same sign as

$$h^{n+1}[\phi^{n+1}(a) - \psi^{n+1}(a)].$$

Hence, if n be odd, $y_1 - y_2$ does not change sign when h is changed into $-h$, and thus the two curves do not cross each other at the common point. On the other hand, if n be even, $y_1 - y_2$ changes sign with h; and therefore when the contact is of even order the curves cross each other at their common point.

For example, the tangent line usually lies entirely on one side of the curve, but at a point of inflexion the tangent crosses the curve.

Again, the circle of second-order contact crosses the curve except at the special points noted later, in which the circle has contact of the third order.

EXERCISES

1. Find the order of contact of the curves First

$$4y = x^2 \text{ and } y = x - 1.$$

2. Find the order of contact of the curves Do not touch, can solve simultaneou

$$x = y^3 \text{ and } x + y + 1 = 0.$$

3. Find the order of contact of the curves

$$4y = x^2 - 4 \text{ and } x^2 - 2y = 3 - y^2.$$

4. Determine the parabola having its axis parallel to the y-axis, which has the closest possible contact with the curve $a^2y = x^3$ at the point (a, a).

5. Determine a straight line which has contact of the second order with the curve

$$y = x^3 - 3x^2 - 9x + 9.$$

6. Find the order of contact of

$$y = \log(x-1) \quad \text{and} \quad x^2 - 6x + 2y + 8 = 0$$

at the point (2, 0).

7. What must be the value of a in order that the curves

$$y = x + 1 + a(x-1)^2 \quad \text{and} \quad xy = 3x - 1$$

may have contact of the second order?

103. Circle of curvature. The circle that has contact of the closest order with a given curve at a specified point is called the *osculating circle* or circle of curvature of the curve at the given point. The radius of this circle is called the radius of curvature, and its center is called the center of curvature at the assigned point.

104. Length of radius of curvature; coördinates of center of curvature. Let the equation of a circle be

$$(X-\alpha)^2 + (Y-\beta)^2 = R^2, \qquad (1)$$

in which R is the radius, and α, β are the coördinates of the center, the current coördinates being denoted by X, Y to distinguish them from the coördinates of a point on the given curve.

It is required to determine R, α, β, so that this circle may have contact of the second order with the given curve at the point (x, y).

From (1), by successive differentiation, it follows that

$$\left.\begin{aligned} (X-\alpha) + (Y-\beta)\frac{dY}{dX} &= 0,\\ 1 + \left(\frac{dY}{dX}\right)^2 + (Y-\beta)\frac{d^2Y}{dX^2} &= 0. \end{aligned}\right\} \qquad (2)$$

If the circle (1) has contact of the second order at the point (x, y) with the given curve, then when $X = x$ it is necessary that

$$\left.\begin{aligned} &Y = y, \\ &\frac{dY}{dX} = \frac{dy}{dx}, \quad \frac{d^2Y}{dX^2} = \frac{d^2y}{dx^2}. \end{aligned}\right\} \qquad (3)$$

Substituting these expressions in (2),

$$\left.\begin{aligned} (x-\alpha) + (y-\beta)\frac{dy}{dx} &= 0, \\ 1 + \left(\frac{dy}{dx}\right)^2 + (y-\beta)\frac{d^2y}{dx^2} &= 0, \end{aligned}\right\} \qquad (4)$$

whence

$$y - \beta = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \quad x - \alpha = \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}, \qquad (5)$$

and finally, by substitution in (1),

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \qquad (6)$$

105. Direction of radius of curvature. Since, at any point P on the given curve, the value of $\frac{dy}{dx}$ is the same for the curve and the osculating circle for that point, it follows that they have the same tangent and normal at P, and hence that the radius of curvature coincides with the normal. Again, since the value of $\frac{d^2y}{dx^2}$ is the same for both curves at P, it follows from Art. 97, that they have the same direction of bending at that point, and hence that the center of curvature lies on the concave side of the given curve (Fig. 44).

It follows from this fact and Art. 102 that the osculating circle is the limiting position of a circle passing through three points on the curve when these points move into coincidence.

The radius of curvature is usually regarded as positive or negative according as the bending of the curve is positive or negative (Art. 97), that is, according as the value of $\frac{d^2y}{dx^2}$ is positive or negative; hence, in the expression for R, the radical in the numerator is always to be given the positive sign. The sign of R changes as the point P passes through a point of inflexion on the given curve (Fig. 45). It is evident from the figure that in this case R passes through an infinite value; for the circle through the points N, P, Q approaches coincidence with the inflexional tangent when N and Q approach coincidence with P, and the center of this circle at the same time passes to infinity.

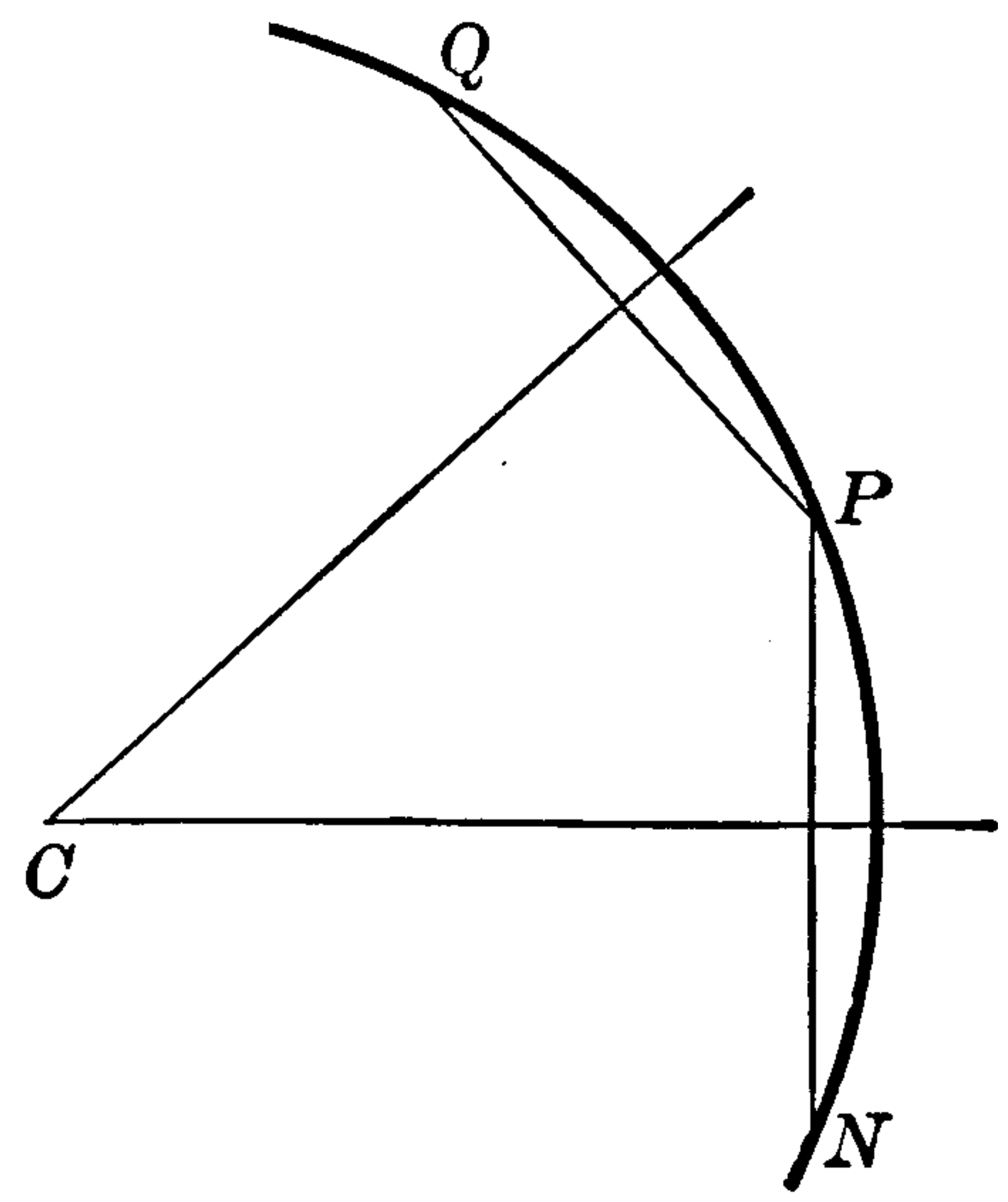


FIG. 44.

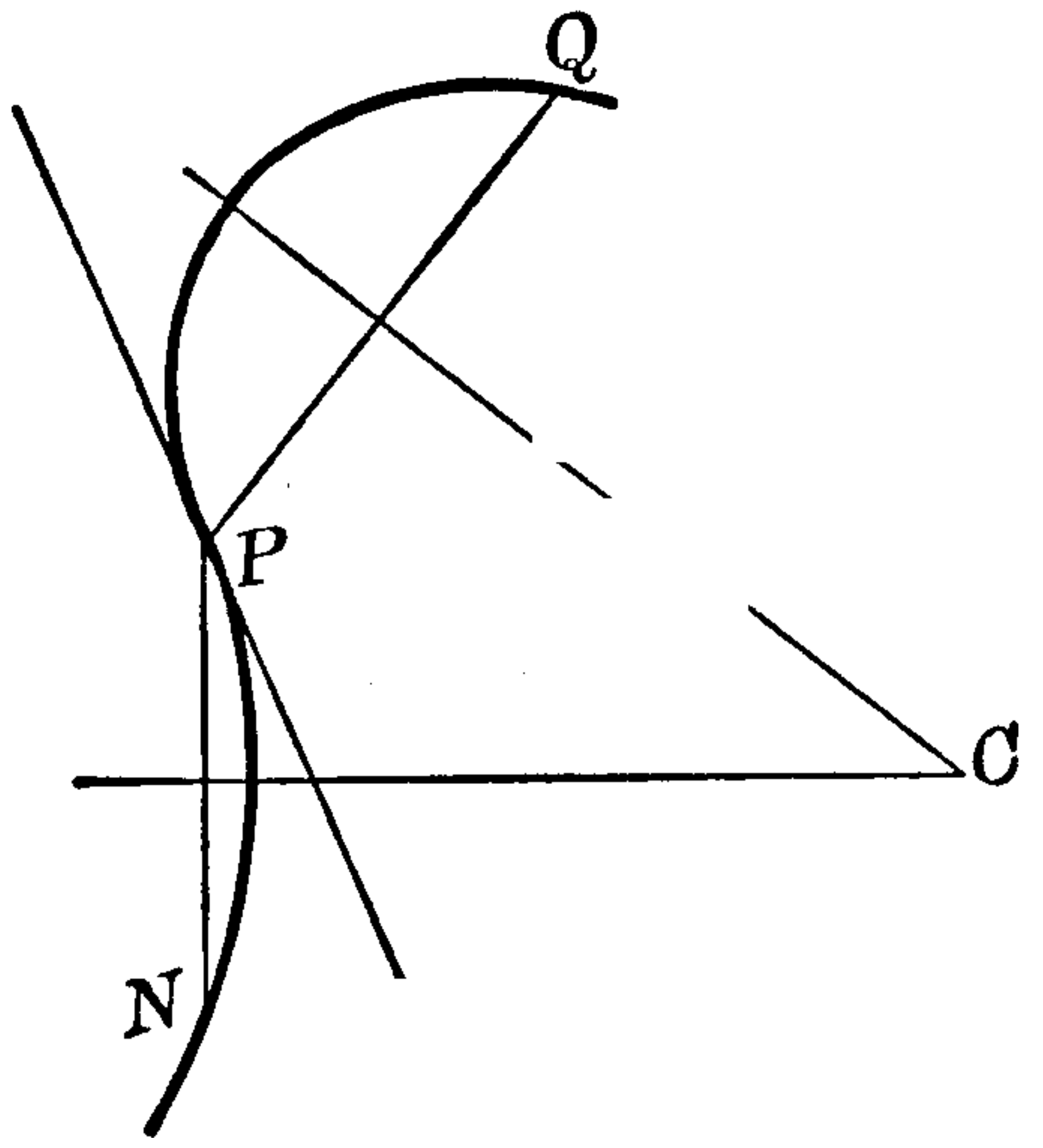


FIG. 45.

106. Total curvature of a given arc; average curvature. The total curvature of an arc PQ (Fig. 46) in which the bending is continuous and in one direction, is the angle through which the tangent swings as the point of contact moves from the initial point P to the terminal point Q; or, in other words, it is the angle between the tangents at P and Q, measured from the former to the latter. Thus the total curvature of a given arc is positive or negative according as the bending is in the positive or negative direction of rotation.

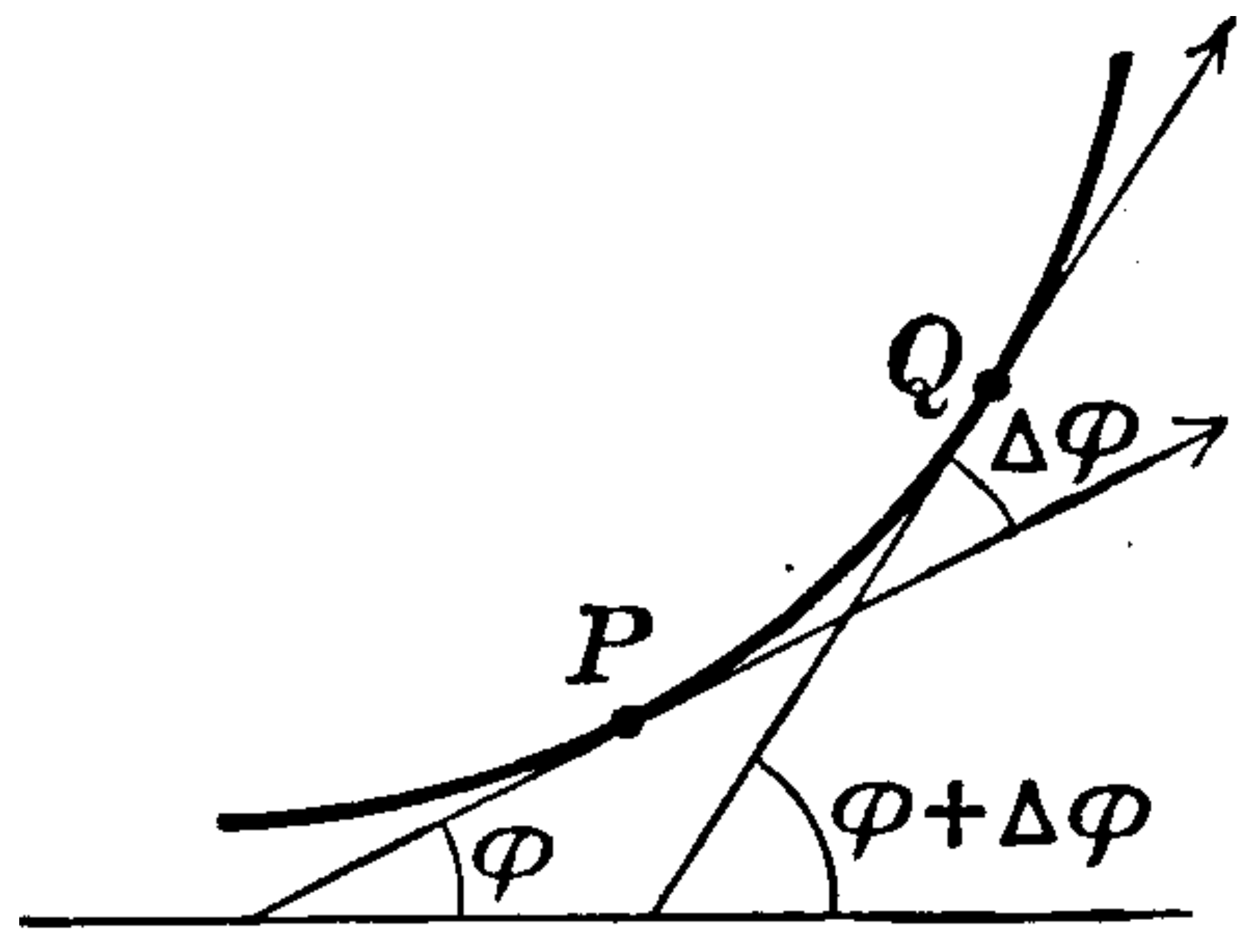


FIG. 46.

The total curvature of an arc divided by the length of the arc is called the *average* curvature of the arc. Thus, if the length of the arc PQ be Δs centimeters, and if its total curvature be $\Delta\phi$ radians, then its average curvature is $\frac{\Delta\phi}{\Delta s}$ radians per centimeter.

107. Measure of curvature at a given point. The *measure of the curvature* of a given curve at a given point P is the limit which the average curvature of the arc PQ approaches when the point Q approaches coincidence with P.

Since the average curvature of the arc PQ is $\frac{\Delta\phi}{\Delta s}$, the measure of the curvature at the point P is

$$\kappa = \lim_{\Delta s \doteq 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds},$$

and may be regarded as the rate of deflection of the arc from the tangent estimated per unit of length; or again, as the ratio of the angular velocity of the tangent to the linear velocity of the point of contact.

To express κ in terms of x, y, and their derivatives, observe

that $$\tan\phi = \frac{dy}{dx}.$$

Whence $$\phi = \tan^{-1}\frac{dy}{dx},$$

and $$\frac{d\phi}{ds} = \frac{d}{ds}\left(\tan^{-1}\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\tan^{-1}\frac{dy}{dx}\right)\cdot\frac{dx}{ds}$$

$$= \frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}\cdot\frac{1}{\dfrac{ds}{dx}};$$

therefore $$\kappa = \frac{d\phi}{ds} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$ [Art. 83

108. Curvature of osculating circle. A curve and its osculating circle at P have the same measure of curvature at that point.

For, let κ, κ' be their respective measures of curvature at the point of contact (x, y). Then from Art. 107,

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

But this is the reciprocal of the expression for the radius of curvature (Eq. (6), p. 164); hence

$$\kappa = \frac{1}{R}.$$

That is: *the measure of curvature κ at a point P is the reciprocal of the radius of curvature R for that point.* Since a curve and its osculating circle have the same radius at their point of contact, it follows from this result that the measure of curvature is also the same for both.

It is on account of this property that the osculating circle is called the circle of curvature. This is sometimes used as the defining property of the circle of curvature. The radius of curvature at P would then be defined as the radius of the circle whose measure of curvature is the same as that of the given curve at the point P. Its value, as found from Art. 106 and Art. 107, accords with that given in Art. 104.

EXERCISES

1. Find the radius of curvature of the curve $y^2 = 4ax$ at the origin.

2. Find the radius of curvature of the curve $y^3+x^3+a(x^2+y^2)=a^2y$ at the origin.

3. Find the radius of curvature of the curve $a^3y = bx^3 + cx^2y$ at the origin.

Find the radius of curvature for each of the following curves:

4. $xy = m^2$. Rectangular hyperbola.

5. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Hyperbola.

6 $a^{n-1}y = x^n$. General parabola.

7. $\sqrt{x}+\sqrt{y}=\sqrt{a}$. Parabola.

8. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. Hypocycloid.

9. $y^2=\frac{x^3}{2a-x}$. Cissoid.

10. $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. Catenary.

109. Direct derivation of the expressions for κ and R in polar coördinates. Using the notation of Art. 81,

$$\phi = \theta + \psi,$$

hence
$$\kappa = \frac{d\phi}{ds} = \frac{\dfrac{d\phi}{d\theta}}{\dfrac{ds}{d\theta}} = \frac{\left(1 + \dfrac{d\psi}{d\theta}\right)}{\dfrac{ds}{d\theta}} \tag{1}$$

$$= \frac{\left(1 + \dfrac{d\psi}{d\theta}\right)}{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}}}. \qquad \text{[Art. 87.}$$

But
$$\tan\psi = \rho\frac{d\theta}{d\rho}, \quad \psi = \tan^{-1}\left[\frac{\rho}{\dfrac{d\rho}{d\theta}}\right],$$

therefore, by differentiating as to θ and reducing,

$$\frac{d\psi}{d\theta} = \frac{\left(\dfrac{d\rho}{d\theta}\right)^2 - \rho\dfrac{d^2\rho}{d\theta^2}}{\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2},$$

which, substituted in (1), gives

$$\kappa = \frac{\rho^2 - \rho\dfrac{d^2\rho}{d\theta^2} + 2\left(\dfrac{d\rho}{d\theta}\right)^2}{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Since $\kappa = \dfrac{1}{R}$, it follows that

$$R = \frac{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 - \rho\dfrac{d^2\rho}{d\theta^2} + 2\left(\dfrac{d\rho}{d\theta}\right)^2}.$$

When $u = \frac{1}{\rho}$ is taken as dependent variable, the expression for κ assumes the simpler form

$$\kappa = \frac{u^3\left(u + \frac{d^2u}{d\theta^2}\right)}{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Since at a point of inflexion κ vanishes and changes sign, hence the condition for a point of inflexion, expressed in polar coördinates, is that $u + \frac{d^2u}{d\theta^2}$ shall vanish and change sign.

EXERCISES

Find the radius of curvature for each of the following curves:

1. $\rho = a^\theta$.
2. $\rho^2 = a^2 \cos 2\theta$.
3. $\rho = 2a\cos\theta - a$.
4. $\rho \cos^2 \frac{1}{2}\theta = a$.
5. $\rho^2 \cos 2\theta = a^2$.
6. $\rho = 2a(1 - \cos\theta)$.
7. $\rho\theta = a$.

EVOLUTES AND INVOLUTES

110. Definition of an evolute. When the point P moves along the given curve, the center of curvature C describes another curve which is called the *evolute* of the first.

Let $f(x, y) = 0$ be the equation of the given curve. Then the equation of the locus described by the point C is found by eliminating x and y from the three equations

$$f(x, y) = 0,$$

$$x - \alpha = \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}},$$

$$y - \beta = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}},$$

and thus obtaining a relation between α, β, the coördinates of the center of curvature.

No general process of elimination can be given; the method to be adopted depends upon the form of the given equation $f(x, y) = 0$.

Ex. 1. Find the evolute of the parabola $y^2 = 4px$.

Since $\quad y = 2p^{\frac{1}{2}}x^{\frac{1}{2}}, \ \dfrac{dy}{dx} = p^{\frac{1}{2}}x^{-\frac{1}{2}}, \ \dfrac{d^2y}{dx^2} = -\dfrac{1}{2}p^{\frac{1}{2}}x^{-\frac{3}{2}},$

hence $\quad x - \alpha = -p^{\frac{1}{2}}x^{-\frac{1}{2}}(1 + px^{-1})\,2p^{-\frac{1}{2}}x^{\frac{3}{2}} = -2(x + p),$

and $\quad y - \beta = (1 + px^{-1})\,2p^{-\frac{1}{2}}x^{\frac{3}{2}} = 2(p^{-\frac{1}{2}}x^{\frac{3}{2}} + p^{\frac{1}{2}}x^{\frac{1}{2}});$

therefore $\quad \alpha = 2p + 3x, \qquad \beta = -2p^{-\frac{1}{2}}x^{\frac{3}{2}}.$

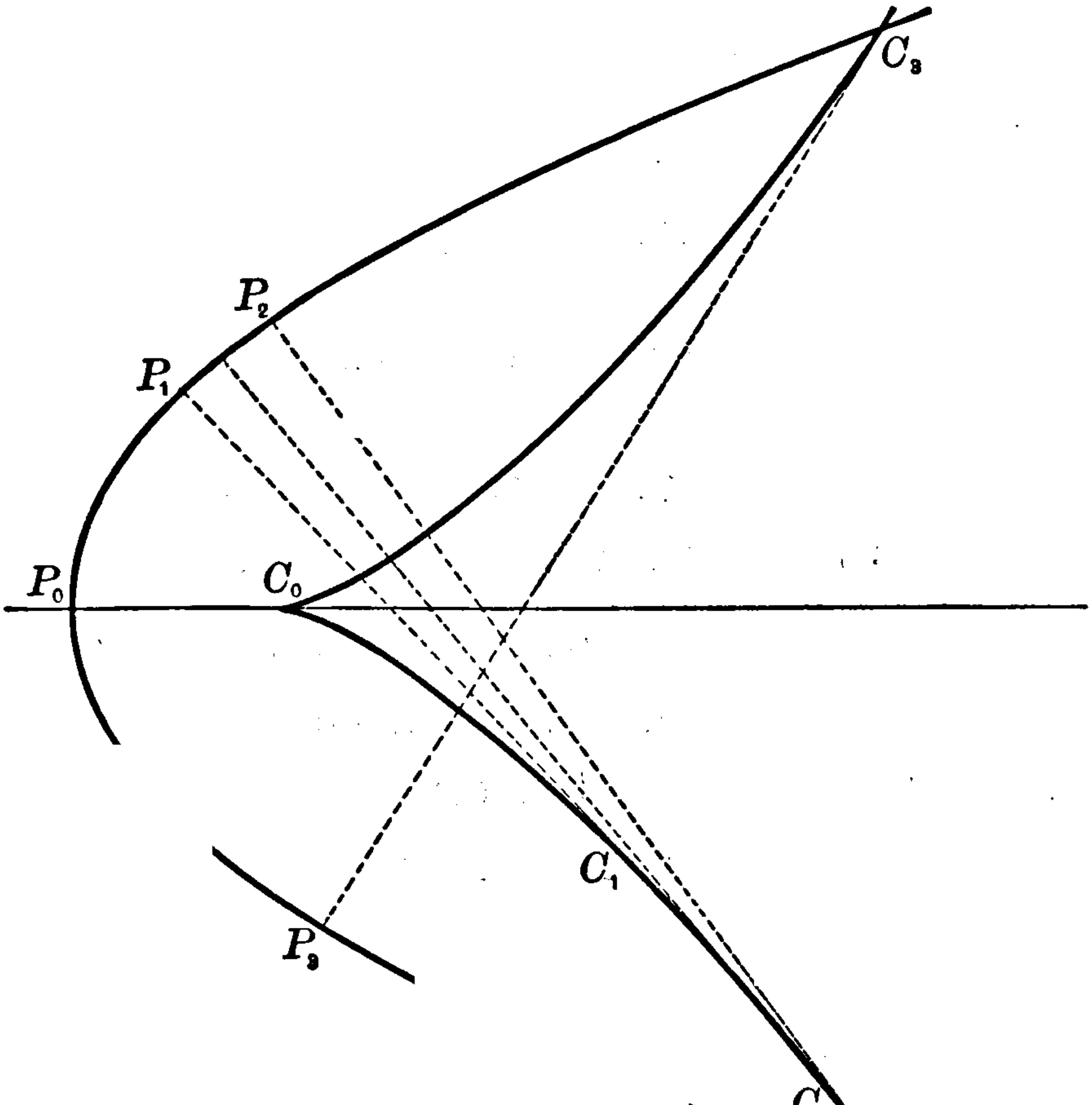


FIG. 47.

The result of eliminating x between the last two equations is

$$\tfrac{1}{27}(\alpha - 2p)^3 = \tfrac{1}{4}(p^{\frac{1}{2}}\beta)^2,$$

i.e., $$4(\alpha - 2p)^3 = 27p\beta^2,$$

which is the equation of the evolute of the parabola, α, β being the current coördinates.

Ex. 2. Find the evolute of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{1}$$

Here $$\frac{x}{a^2}+\frac{y}{b^2}\cdot\frac{dy}{dx}=0,\quad \frac{dy}{dx}=-\frac{b^2x}{a^2y},$$

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\frac{y-x\dfrac{dy}{dx}}{y^2}=\frac{-b^2}{a^2y^2}\left(y+\frac{b^2x^2}{a^2y}\right)=\frac{-b^2}{a^4y^3}(a^2y^2+b^2x^2)=\frac{-b^4}{a^2y^3},$$

whence

$$y-\beta=\frac{(a^4y^2+b^4x^2)y}{a^2b^4}=\left(\frac{a^2y^2}{b^4}+\frac{x^2}{a^2}\right)y=\left(\frac{a^2y^2}{b^4}+1-\frac{y^2}{b^2}\right)y.$$

Therefore $$-\beta=\frac{a^2-b^2}{b^4}y^3. \tag{2}$$

Similarly, $$\alpha=\frac{a^2-b^2}{a^4}x^3. \tag{3}$$

Eliminating x, y between (1), (2), (3), the equation of the locus described by (α, β) is

$$(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}. \quad \text{(Fig. 52)}$$

111. Properties of the evolute. The evolute has two important properties that will now be established.

I. *The normal to the curve is tangent to the evolute.* The relations connecting the coördinates (α, β) of the center of curvature with the coördinates (x, y) of the corresponding point on the curve are, by Art. 104,

$$x-\alpha+(y-\beta)\frac{dy}{dx}=0, \tag{1}$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-\beta)\frac{d^2y}{dx^2}=0. \tag{2}$$

By differentiating (1) as to x, considering α, β, y as functions of x,

$$1+\left(\frac{dy}{dx}\right)^2+(y-\beta)\frac{d^2y}{dx^2}-\frac{d\alpha}{dx}-\frac{d\beta}{dx}\frac{dy}{dx}=0. \tag{3}$$

Subtracting (3) from (2),

$$\frac{d\alpha}{dx}+\frac{d\beta}{dx}\frac{dy}{dx}=0, \tag{4}$$

whence $$\frac{d\beta}{d\alpha}=-\frac{dx}{dy}.$$

But $\frac{d\beta}{d\alpha}$ is the slope of the tangent to the evolute at (α, β), and $-\frac{dx}{dy}$ is the slope of the normal to the given curve at (x, y). Hence these lines have the same slope; but they pass through the same point (α, β), therefore they are coincident.

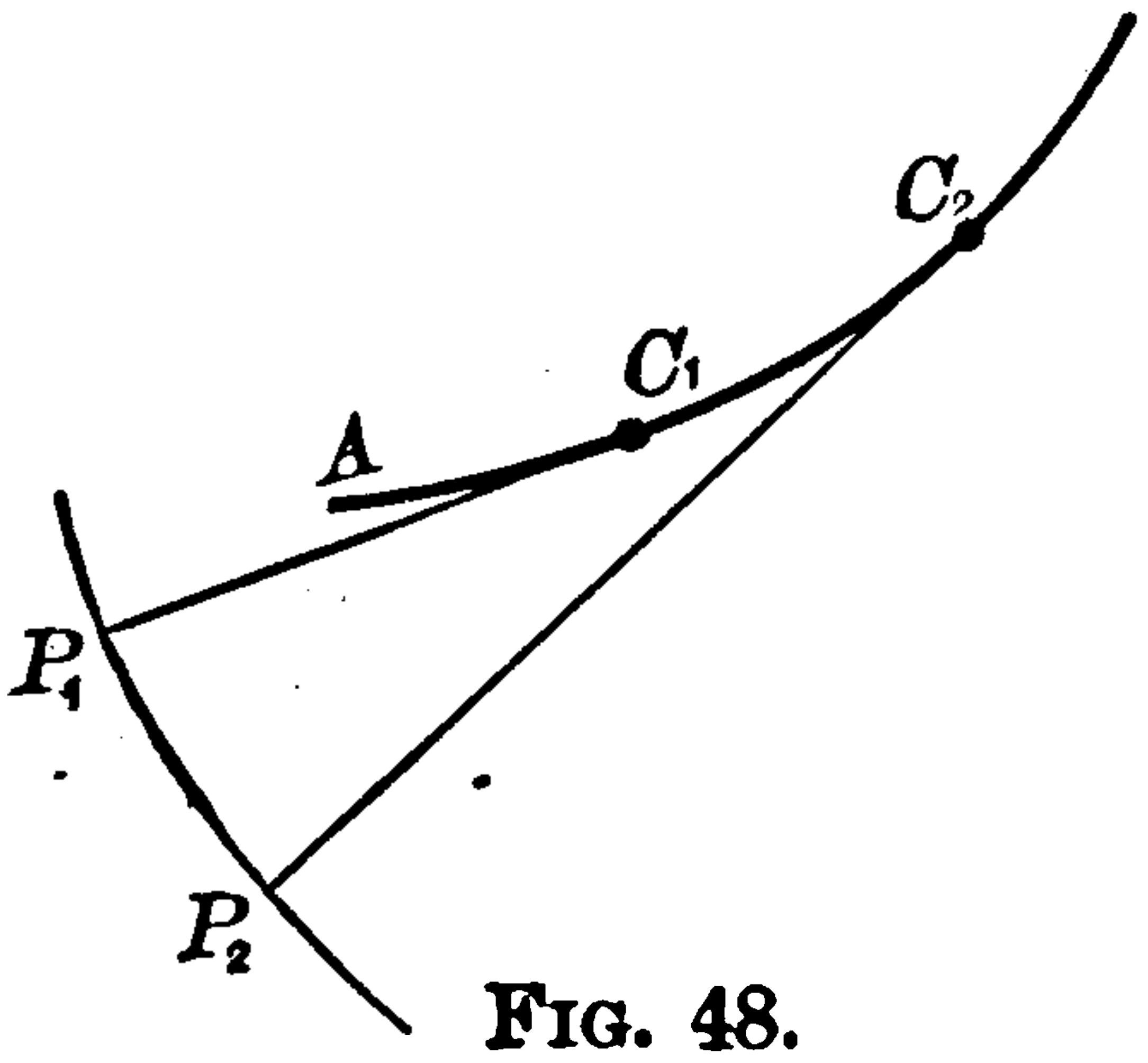


FIG. 48.

II. *The difference between two radii of curvature of the given curve, which touch the evolute at the points* C_1, C_2 *(Fig. 48), is equal to the arc* C_1C_2 *of the evolute.*

Since R is the distance between points (x, y), (α, β), hence

$$(x-\alpha)^2+(y-\beta)^2=R^2. \tag{5}$$

When the point (x, y) moves along the given curve, the point (α, β) moves along the evolute, and thus α, β, R, y are all functions of x.

Differentiation of (5) as to x gives

$$(x-\alpha)\left(1-\frac{d\alpha}{dx}\right)+(y-\beta)\left(\frac{dy}{dx}-\frac{d\beta}{dx}\right)=R\frac{dR}{dx}; \tag{6}$$

hence, subtracting (6) from (1),

$$(x-\alpha)\frac{d\alpha}{dx}+(y-\beta)\frac{d\beta}{dx}=-R\frac{dR}{dx}. \tag{7}$$

Again, from (1) and (4),

$$\frac{\frac{d\alpha}{dx}}{x-\alpha}=\frac{\frac{d\beta}{dx}}{y-\beta}. \tag{8}$$

Hence, each of these fractions is equal to

$$\frac{\sqrt{\left(\frac{d\alpha}{dx}\right)^2+\left(\frac{d\beta}{dx}\right)^2}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}}=\pm\frac{\frac{d\sigma}{dx}}{R}, \tag{9}$$

in which σ is the arc of the evolute. (Compare Art. 64.)

Next, multiplying numerator and denominator of the first member of (8) by $x-\alpha$, and those of the second member by $y-\beta$, and combining new numerators and denominators, it follows that each of the fractions in (8) is equal to

$$\frac{(x-\alpha)\frac{d\alpha}{dx}+(y-\beta)\frac{d\beta}{dx}}{(x-\alpha)^2+(y-\beta)^2},$$

which equals $-\frac{R\frac{dR}{dx}}{R^2}$ by (7) and (5).

By combining with (9),

$$\frac{d\sigma}{dx}=\pm\frac{dR}{dx},$$

that is, $$\frac{d}{dx}(\sigma\pm R)=0.$$

Therefore $$\sigma\pm R=\text{constant}, \tag{10}$$

wherein σ is measured from a fixed point A on the evolute.

Now, let C_1, C_2 be the centers of curvature for the points

P_1, P_2 on the given curve; let $P_1C_1 = R_1$, $P_2C_2 = R_2$; and let the arcs AC_1, AC_2 be denoted by σ_1, σ_2. Then

$$\sigma_1 \pm R_1 = \sigma_2 \pm R_2, \text{ by } (10);$$

that is,
$$\sigma_1 - \sigma_2 = \pm (R_2 - R_1);$$

hence,
$$\text{arc } C_1C_2 = R_2 - R_1. \tag{11}$$

Thus, in Fig. 49,

$$P_1C_1 + C_1C_2 = P_2C_2,$$
$$P_2C_2 + C_2C_3 = P_3C_3, \text{ etc.}$$

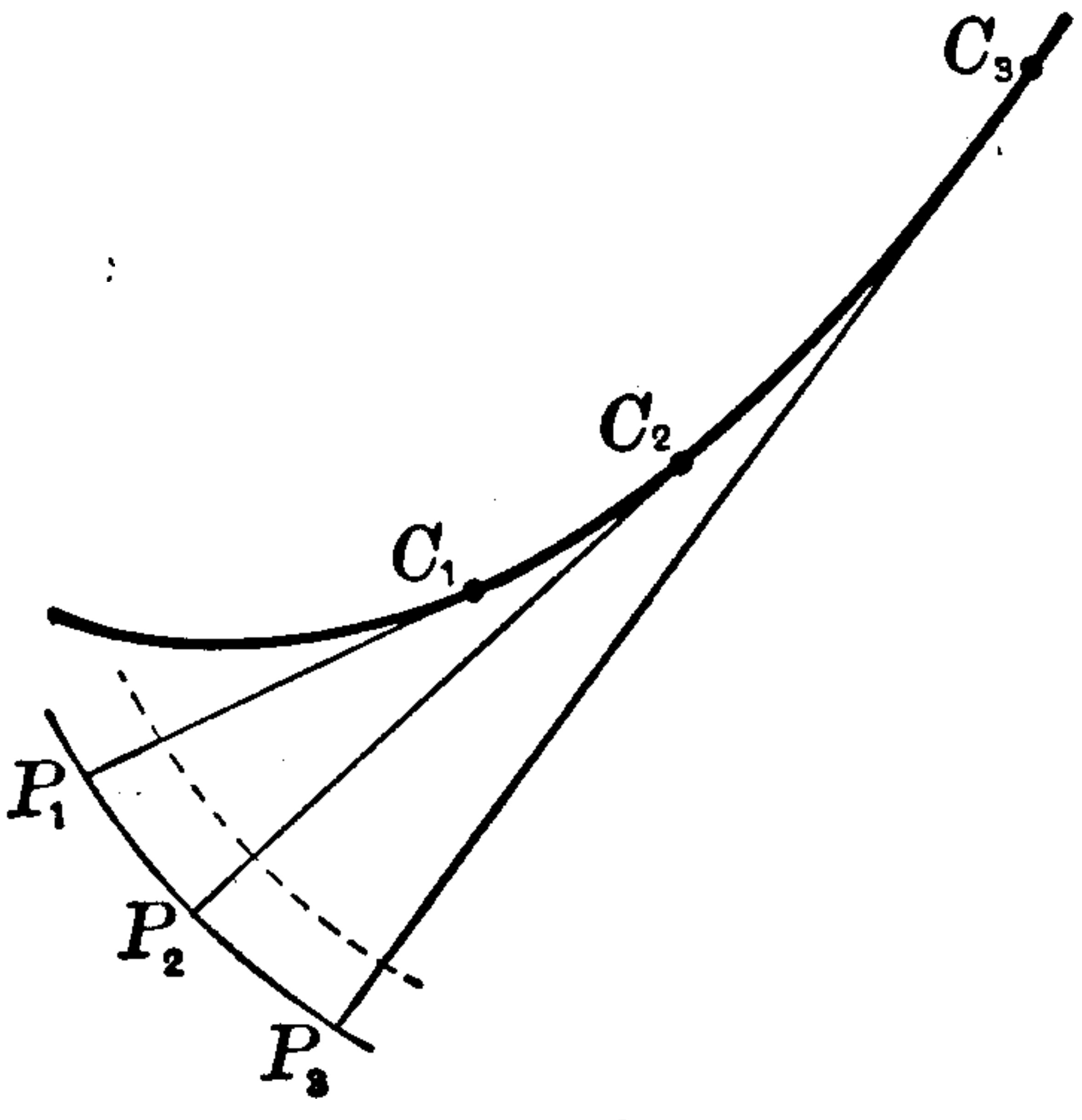


FIG. 49.

Hence, if a thread be wrapped around the evolute, and then be unwound, the free end of it can be made to trace out the original curve. From this property the locus of the centers of curvature of a given curve is called the *evolute* of that curve, and the latter is called the *involute* of the former.

When the string is unwound, each point of it describes a different involute; hence, any curve has an infinite number of involutes, but only one evolute.

Any two of these involutes intercept a constant distance on their common normal, and are called *parallel* curves on account of this property.

Ex. Find the length of that part of the evolute of the parabola which lies inside the curve.

From Fig. 47 the required length is twice the difference between the tangents C_3P_3 and P_0C_0, both of which are normals to the parabola.

To find the coördinates of the point P_3, write the equation of the tangent to the evolute at C_3, and find the other point at which it intersects the parabola.

The coördinates of C_3, the point of intersection of the two curves, are $(8p;\ 4p\sqrt{2})$, and the equation of the tangent at C_3 is

$$2x - \sqrt{2}\,y - 8p = 0.$$

This tangent intersects the parabola at the point $(2p, -2\sqrt{2}p)$, which is P_3.

The value of the radius of curvature is $\frac{2(x+p)^{\frac{3}{2}}}{\sqrt{p}}$, hence $P_0C_0 = 2p$, $P_3C_3 = 6\sqrt{3}p$, hence the arc C_0C_3 is $2p(3\sqrt{3}-1)$, and the required length of the evolute is therefore $4p(3\sqrt{3}-1)$.

EXERCISES

Find the coördinates of the center of curvature for each of the following curves:

✓ **1.** $x^2 + y^2 = a^2$.

2. $x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$.

✓ **3.** $y^3 = a^2 x$.

4. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

Find the equations of the evolutes of the following curves:

✓ **5.** $xy = a^2$.

✓ **6.** $a^2y^2 - b^2x^2 = -a^2b^2$.

✓ **7.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

8. Show that the curvature of an ellipse is a minimum at the end of the minor axis, and that the osculating circle at this point has contact of the third order with the curve.

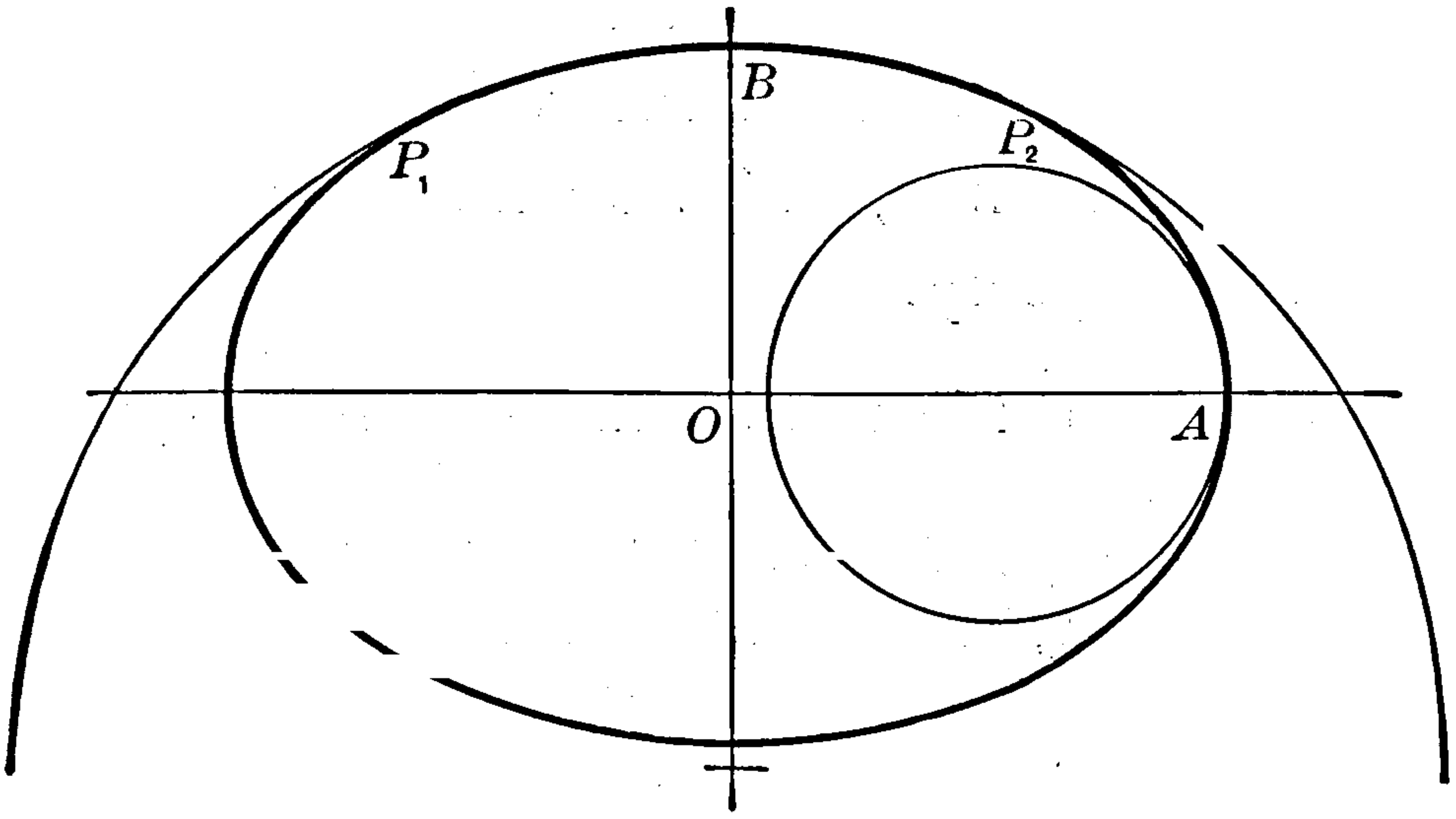


FIG. 50.

This circle of curvature must be entirely outside the ellipse (Fig. 50). For, consider two points P_1, P_2, one on each side of B, the end of the minor axis. At these points the curvature is greater than at B, hence these points must be farther from the tangent at B than the circle of curvature, which has everywhere the same curvature as at B.

9. Similarly, show that the curvature at A, the end of the major axis, is a maximum, and that the circle of curvature at A lies entirely within the ellipse (Fig. 50).

10. Show how to sketch the circle of curvature for points between A and B. The circle of curvature for points between A and B has three coincident points in common with the ellipse (Art. 104), hence the circle crosses the curve (Art. 102). Let K, P, L be three points on the arc, such that K is nearest A and L nearest B. The center of curvature for P lies on the normal to P, and on the concave side of the curve. The circle crosses at P, lying outside of the ellipse at K (on the side towards A), and inside the ellipse at L; for the bending of the ellipse increases from B to P and from P to K, while the bending (curvature) of the osculating circle remains constant (Fig. 51).

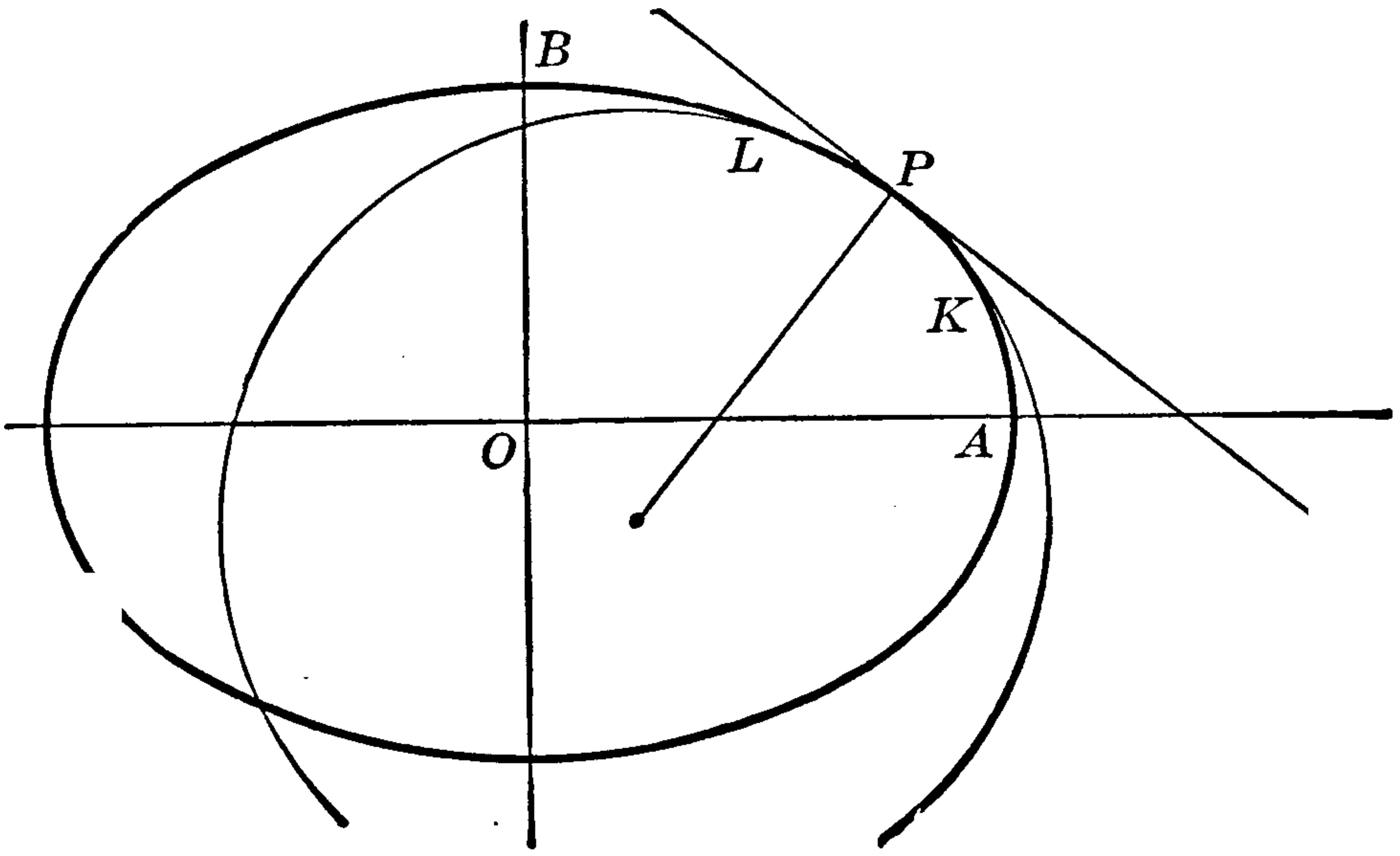


FIG. 51.

11. Two centers of curvature lie on every normal. Prove geometrically that the normals to the curve are tangents to the evolute.

12. Show that the entire length of the evolute of the ellipse is $4\left(\frac{a^2}{b} - \frac{b^2}{a}\right)$. [From equation (11) above, take R_1, R_2 as the radii of curvature at the extremities of the major and minor axes.]

13. If E be the center of curvature at the vertex A (Fig. 52), prove that $CE = ae^2$, in which e is the eccentricity of the ellipse; and hence that CD, CA, CF, CE form a geometric series whose common ratio is e. Show also that DA, AF, FE form a similar series.

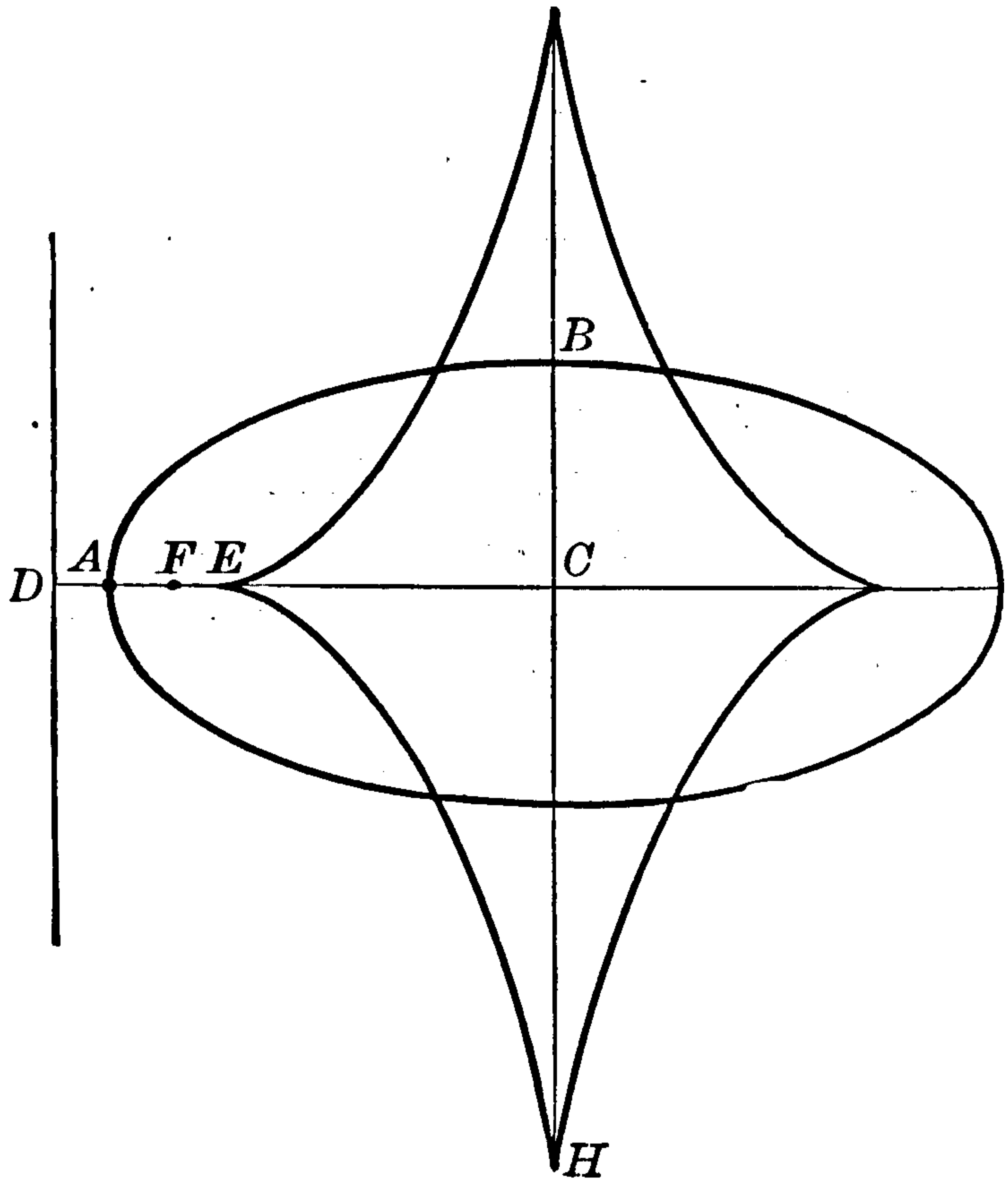


FIG. 52.

14. If H be the center of curvature at B, show that the point H is without or within the ellipse, according as $a >$ or $< b\sqrt{2}$, or according as $e^2 >$ or $< \frac{1}{2}$.

15. Show by inspection of the figure that four real normals can be drawn to the ellipse from any point within the evolute.

CHAPTER XV

SINGULAR POINTS

112. Definition of a singular point. If the equation $f(x, y)=0$ be represented by a curve, the derivative $\frac{dy}{dx}$, when it has a determinate value, measures the slope of the tangent at the point (x, y). There may be certain points on the curve, however, at which the expression for the derivative assumes an illusory or indeterminate form; and, in consequence, the slope of the tangent at such a point cannot be directly determined by the method of Art. 10. Such values of x, y are called *singular values*, and the corresponding points on the curve are called *singular points.*

113. Determination of singular points of algebraic curves. When the equation of the curve is rationalized and cleared of fractions, let it take the form $f(x, y)=0$.

This gives, by differentiation with regard to x, as in Art. 71,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0,$$

whence

$$\frac{dy}{dx}=-\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \tag{1}$$

In order that $\frac{dy}{dx}$ may become illusory, it is therefore necessary that

$$\frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=0. \tag{2}$$

Thus to determine whether a given curve $f(x, y) = 0$ has singular points, put $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ each equal to zero and solve these equations for x and y.

If any pair of values of x and y, so found, satisfy the equation $f(x, y) = 0$, the point determined by them is a singular point on the curve.

To determine the appearance of the curve in the vicinity of a singular point (x_1, y_1), evaluate the indeterminate form

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{0}{0},$$

by finding the limit approached continuously by the slope of the tangent when $x \doteq x_1$, $y \doteq y_1$.

Hence
$$\frac{dy}{dx} = -\frac{\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right)}{\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right)}$$

$$= -\frac{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \, \partial y}\frac{dy}{dx}}{\frac{\partial^2 f}{\partial x \, \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}} \qquad \text{[Arts. 49, 72.}$$

at the point (x_1, y_1).

This equation cleared of fractions gives, to determine the slope at (x_1, y_1), the quadratic

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{\partial^2 f}{\partial x \, \partial y}\left(\frac{dy}{dx}\right) + \frac{\partial^2 f}{\partial x^2} = 0. \tag{3}$$

This quadratic equation has in general two roots. The only exceptions occur when simultaneously, at the point in question,

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial x \, \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \tag{4}$$

in which case $\frac{dy}{dx}$ is still indeterminate in form, and must be evaluated as before. The result of the next evaluation is a cubic in $\frac{dy}{dx}$, which gives three values to the slope, unless all the third partial derivatives vanish simultaneously at the singular point.

The geometric interpretation of the two roots of equation (3) will now be given, and similar principles will apply when the quadratic is replaced by an equation of higher degree.

The two roots of (3) are real and distinct, real and coincident, or imaginary, according as

$$H \equiv \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$$

is positive, zero, or negative. These three cases will be considered separately.

114. Multiple points. First let H be positive. Then at the point (x, y) for which $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, there are two values of the slope, and hence two distinct singular tangents. It follows from this that the curve goes through the point in two directions, or, in other words, two branches of the curve cross at this point. Such a point is called a real double point of the curve, or simply a *node*. The conditions, then, to be satisfied at a node (x_1, y_1) are

$$f(x_1, y_1) = 0, \quad \frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial y_1} = 0,$$

and $$H(x_1, y_1) > 0.$$

Ex. Examine for singular points the curve

$$3x^2 - xy - 2y^2 + x^3 - 8y^3 = 0.$$

Here $\frac{\partial f}{\partial x} = 6x - y + 3x^2, \ \frac{\partial f}{\partial y} = -x - 4y - 24y^2.$

The values $x = 0$, $y = 0$ will satisfy these three equations, hence (0, 0) is a singular point.

Since
$$\frac{\partial^2 f}{\partial x^2} = 6 + 6x = 6 \text{ at } (0, 0),$$
$$\frac{\partial^2 f}{\partial x \, \partial y} = -1,$$
$$\frac{\partial^2 f}{\partial y^2} = -4 - 48y = -4 \text{ at } (0, 0),$$

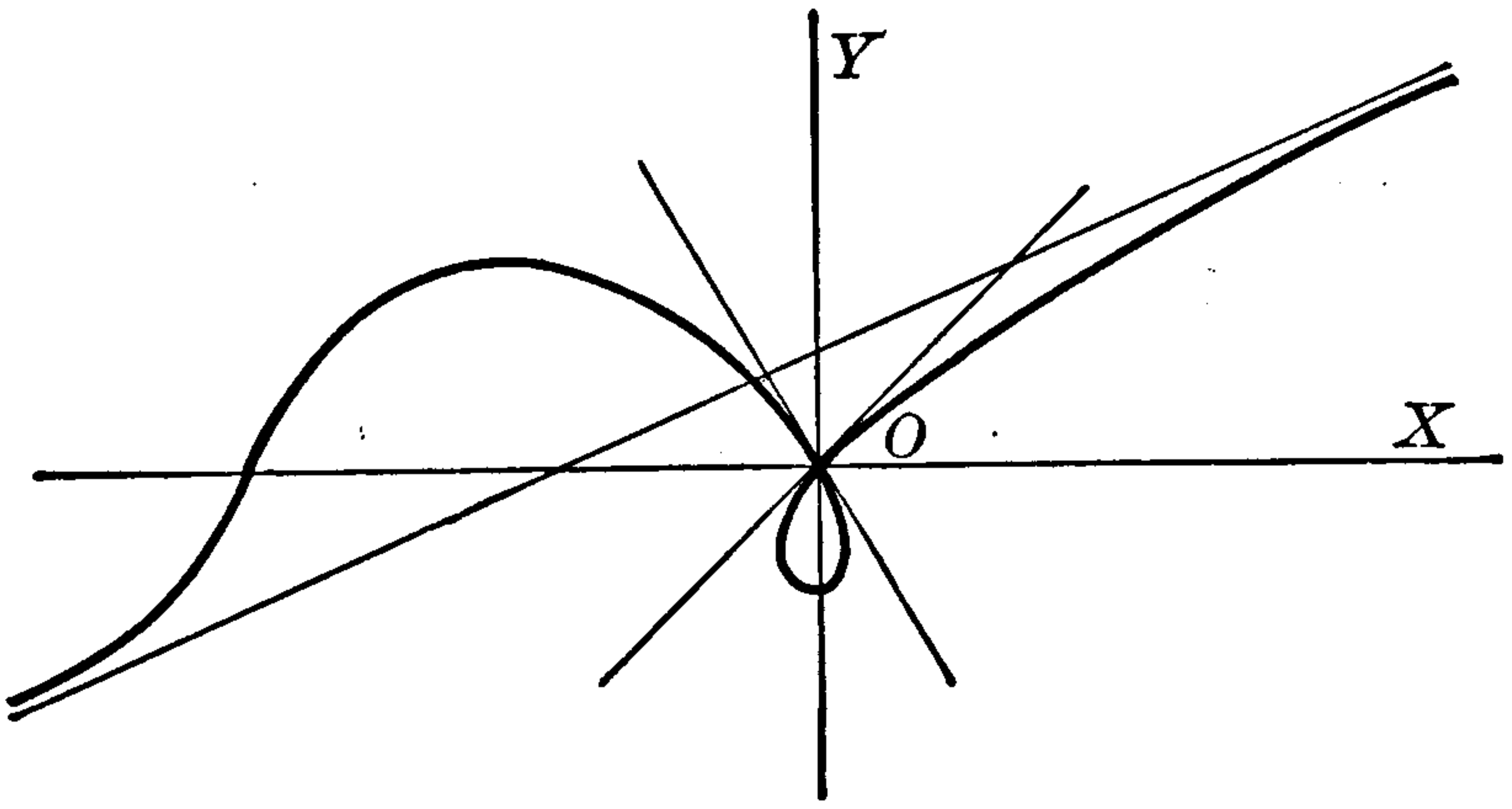


FIG. 53.

hence the equation determining the slope is, from (3),

$$-4\left(\frac{dy}{dx}\right)^2 - 2\left(\frac{dy}{dx}\right) + 6 = 0,$$

of which the roots are 1 and $-\frac{3}{2}$. It follows that (0, 0) is a double point at which the tangents have the slopes 1, $-\frac{3}{2}$.

115. Cusps. Next let $H = 0$. The two tangents are then coincident, and there are two cases to consider. If the curve recedes from the tangent in both directions from the point of tangency, the singular point is called a *tacnode*. Two distinct branches of the curve touch each other at this point. (See Fig. 54.)

If both branches of the curve recede from the tangent in only one direction from the point of tangency, the point is called a *cusp*.

Here again there are two cases to be distinguished. If the branches recede from the point on opposite sides of the double tangent, the cusp is said to be of the first kind; if they recede on the same side, it is called a cusp of the second kind.

The method of investigation will be illustrated by a few examples.

Ex. 1. $$f(x, y) = a^4y^2 - a^2x^4 + x^6 = 0.$$

$$\frac{\partial f}{\partial x} = -4a^2x^3 + 6x^5; \quad \frac{\partial f}{\partial y} = 2a^4y.$$

The point (0, 0) will satisfy $f(x, y) = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$; hence it is a singular point. Proceeding to the second derivatives,

$$\frac{\partial^2 f}{\partial x^2} = -12a^2x^2 + 30x^4 = 0 \text{ at } (0, 0),$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = 2a^4.$$

The two values of $\frac{dy}{dx}$ are therefore coincident, and each equal to zero. From the form of the equation, the curve is evidently symmetrical with regard to both axes; hence the point (0, 0) is a tacnode.

No part of the curve can be at a greater distance from the y-axis than $\pm a$, at which points $\frac{dy}{dx}$ is infinite. The maximum value of y corresponds to $x = \pm a\sqrt{\frac{2}{3}}$. Between $x = 0$, $x = a\sqrt{\frac{2}{3}}$ there is a point of inflexion (Fig. 54).

Ex. 2. $f(x, y) = y^2 - x^3 = 0$;

$$\frac{\partial f}{\partial x} = -3x^2, \quad \frac{\partial f}{\partial y} = 2y.$$

Hence the point (0, 0) is a singular point.

Further, $\frac{\partial^2 f}{\partial x^2} = -6x = 0$ at (0, 0);

$$\frac{\partial^2 f}{\partial x \partial y} = 0; \quad \frac{\partial^2 f}{\partial y^2} = 2.$$

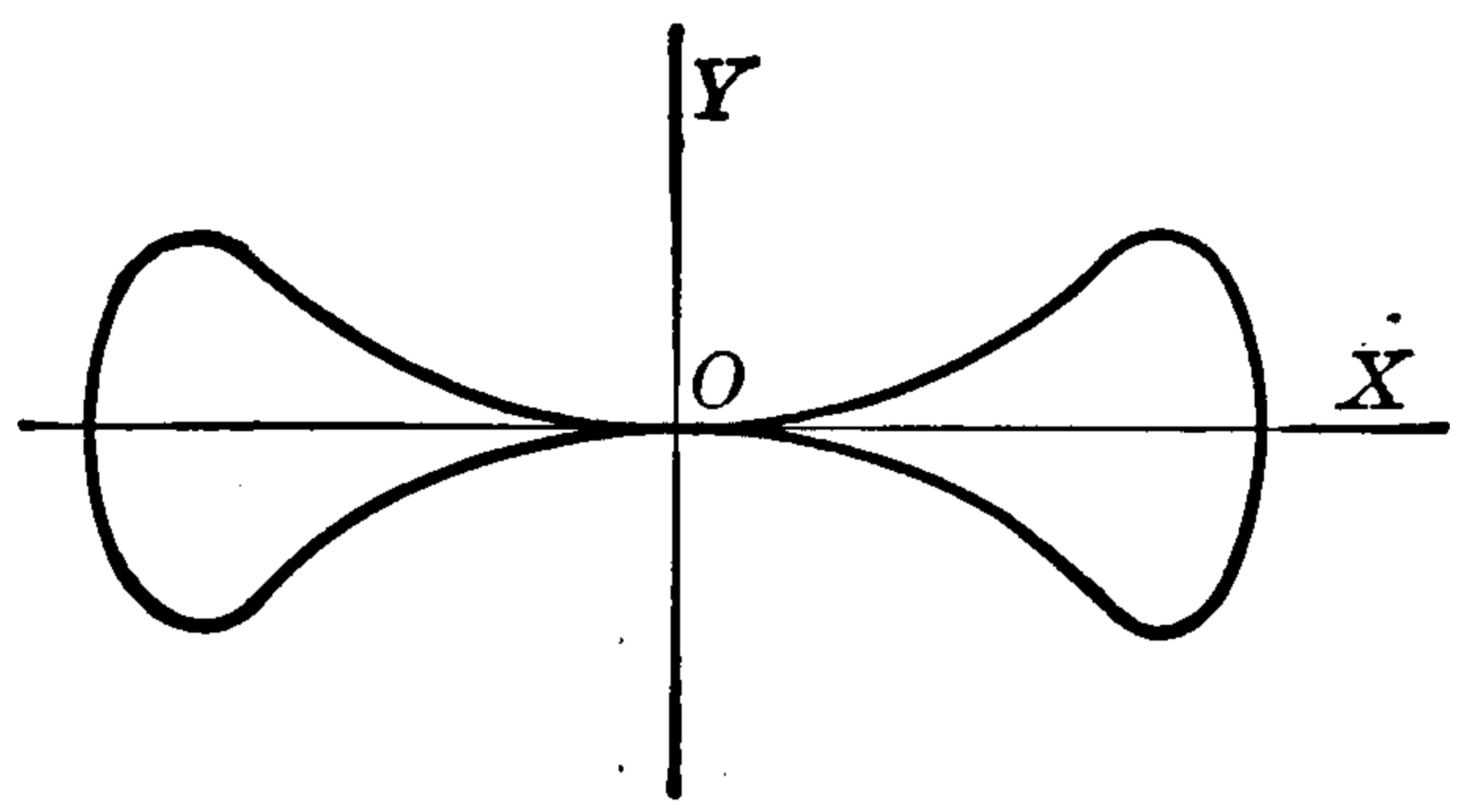


Fig. 54.

Therefore the two roots of the quadratic equation defining $\frac{dy}{dx}$ are both equal to zero. So far, this case is exactly like the last one, but here no part of the curve lies to the left of the axis y. On the right side, the curve is symmetric with regard to the x-axis. As x increases, y increases; there are no maxima nor minima, and no inflexions (Fig. 55).

Ex. 3. $$f(x, y) = x^4 - 2\,ax^2y - axy^2 + a^2y^2 = 0.$$

The point (0, 0) is a singular point, and the roots of the quadratic defining $\frac{dy}{dx}$ are both equal to zero.

Let a be positive. Solving the equation for y,

$$y = \frac{x^2}{a - x}\left(1 \pm \sqrt{\frac{x}{a}}\right).$$

When x is negative, y is imaginary; when $x = 0$, $y = 0$; when x is positive, but less than a, y has two positive values, therefore two branches are above the x-axis. When $x = a$, one branch becomes infinite, having the asymptote $x = a$; the other branch has the ordinate $\frac{1}{2}a$. The origin is therefore a cusp of the second kind (Fig. 56).

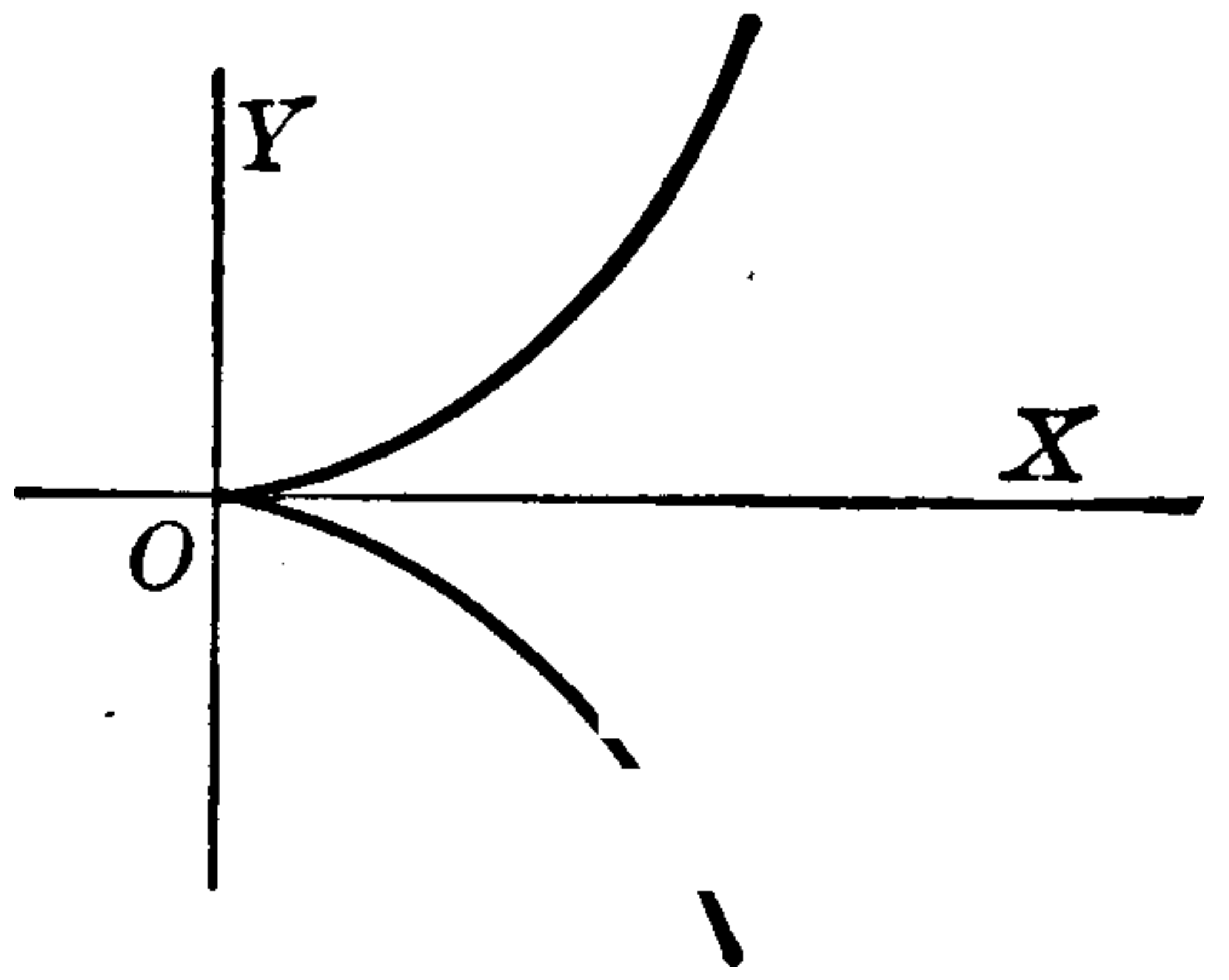


FIG. 55.

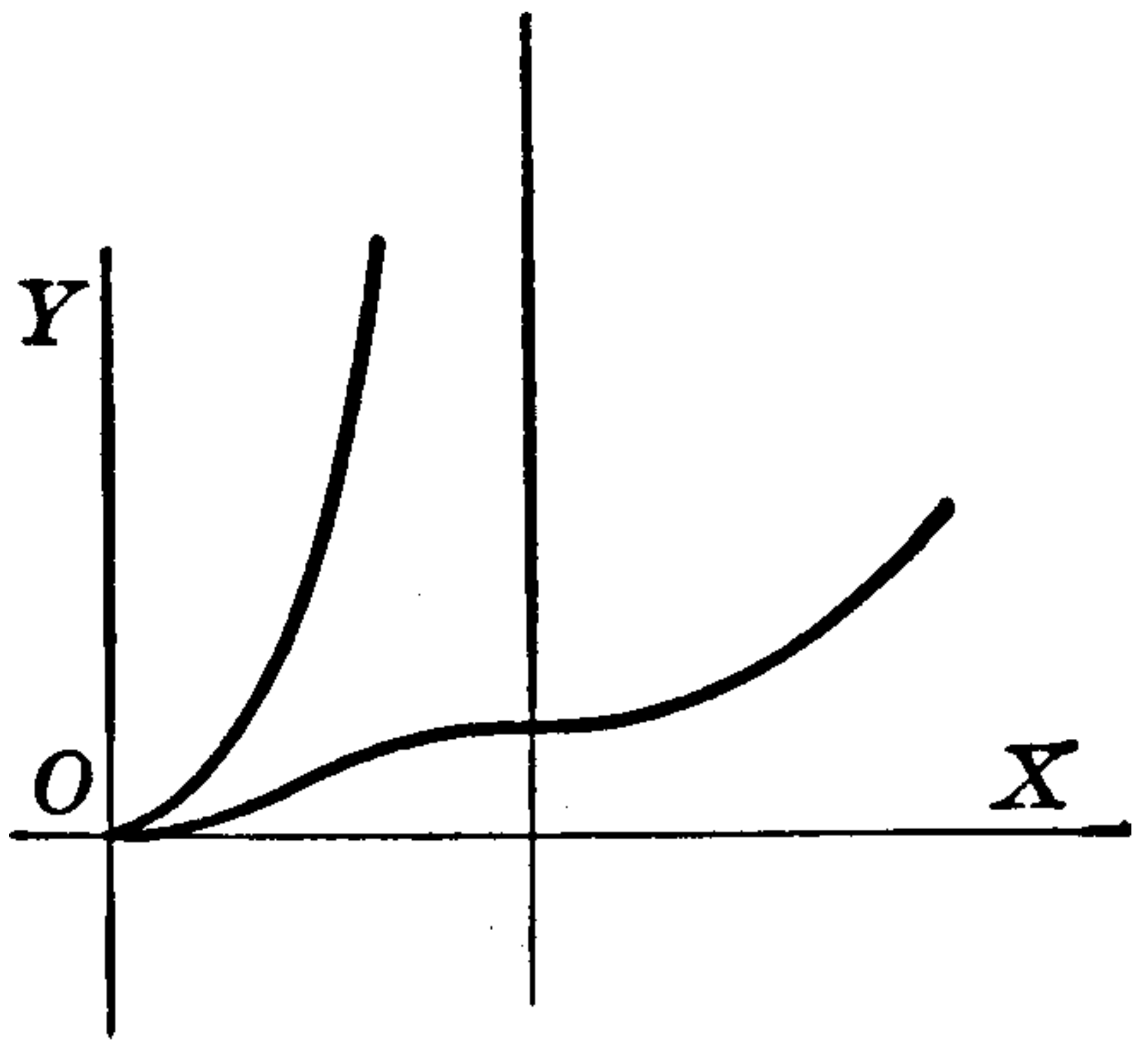


FIG. 56.

116. Conjugate points. Lastly, let H be negative. In this case there are no real tangents; hence no points in the immediate vicinity of the given point satisfy the equation of the curve.

Such an isolated point is called a *conjugate point.*

Ex. $f(x, y) = ay^2 - x^3 + bx^2 = 0.$

Here (0, 0) is a singular point of the locus, and

$$\frac{dy}{dx} = \pm\sqrt{\frac{-b}{a}},$$

both roots being imaginary if a and b have the same sign.

To show the form of the curve, solve the given equation for y.

Then $$y = \pm x\sqrt{\frac{x-b}{a}},$$

and hence, if a and b are positive, there are no real points on the curve between $x = 0$ and $x = b$. Thus O is an isolated point (Fig. 57).

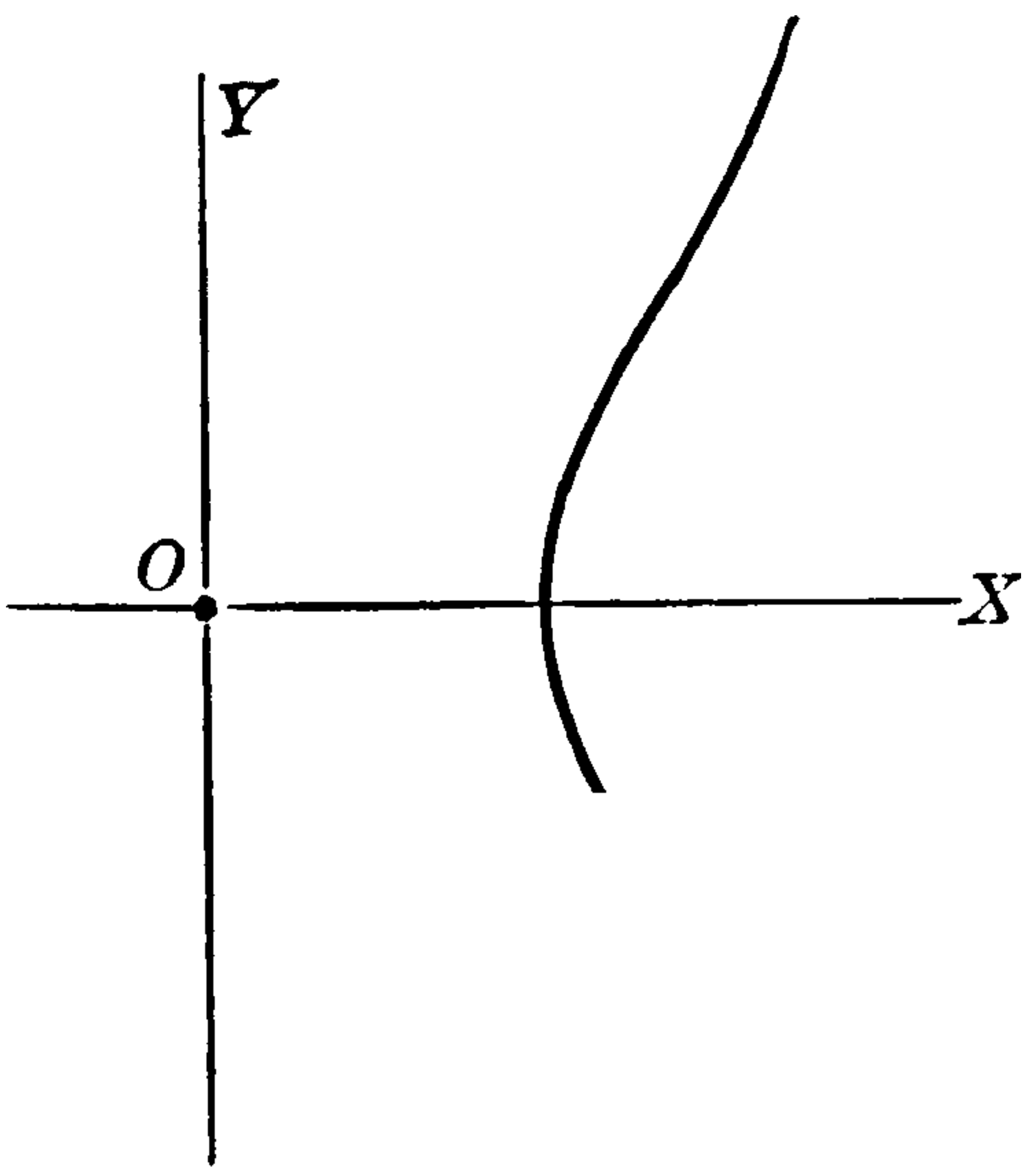


FIG. 57.

These are the only singularities that algebraic curves can have, although complicated combinations of them may appear. In each of the foregoing examples, the singular point was (0, 0); but for any other point, the same reasoning will apply.

Ex. $$f(x, y) = x^2 + 3y^3 - 13y^2 - 4x + 17y - 3 = 0,$$

$$\frac{\partial f}{\partial x} = 2x - 4,\quad \frac{\partial f}{\partial y} = 9y^2 - 26y + 17.$$

At the point (2, 1), $f(2, 1) = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$; hence (2, 1) is a singular point.

Also $\frac{\partial^2 f}{\partial x^2} = 2$; $\frac{\partial^2 f}{\partial x\,\partial y} = 0$; $\frac{\partial^2 f}{\partial y^2} = 18y - 26, = -8$ at (2, 1).

Hence $\frac{dy}{dx} = \pm\frac{1}{2}$; and thus the equations of the two tangents at the node (2, 1) are $y - 1 = \frac{1}{2}(x - 2)$, $y - 1 = -\frac{1}{2}(x - 2)$.

When H is negative, the singular point is necessarily a conjugate point, but the converse is not always true. A singular point may be a conjugate point when $H = 0$. [Compare Ex. 4 below.]

EXERCISES ON CHAPTER XV

Examine each of the following curves for multiple points and find the equations of the tangents at each such point:

1. $a^2x^2 = b^2y^2 + x^2y^2$.

2. $y^2 = \dfrac{x^3}{2a - x}$.

3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

4. $y^2(x^2 - a^2) = x^4$.

5. $y = a + x + bx^2 \pm cx^{\frac{5}{2}}$.

When a curve has two parallel asymptotes it is said to have a node at infinity in the direction of the parallel asymptotes. Apply to Ex. 6.

6. $(x^2 - y^2)^2 - 4y^2 + y = 0$.

7. $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$.

8. $y^2 = x(x + a)^2$.

9. $x^3 - 3axy + y^3 = 0$.

10. $y^2 = x^4 + x^5$.

11. Show that the curve $y = x \log x$ has a terminating point at the origin.

CHAPTER XVI

ENVELOPES

117. Family of curves. The equation of a curve,

$$f(x, y) = 0,$$

usually involves, besides the variables x and y, certain coefficients that serve to fix the size, shape, and position of the curve. The coefficients are called constants with reference to the variables x and y, but it has been seen in previous chapters that they may take different values in different problems, while the form of the equation is preserved. Let α be one of these "constants." Then if α be given a series of numerical values, and if the locus of the equation, corresponding to each special value of α be traced, a series of curves is obtained, all having the same general character, but differing somewhat from each other in size, shape, or position. A system of curves so obtained is called a *family* of curves.

For example, if h, k be fixed, and p be arbitrary, the equation $(y - k)^2 = 2p(x - h)$ represents a family of parabolas, each curve of which has the same vertex (h, k), and the same axis $y = k$, but a different latus rectum. Again, if k be the arbitrary constant, this equation represents a family of parabolas having parallel axes, the same latus rectum, and having their vertices on the same line $x = h$.

The presence of an arbitrary constant α in the equation of a curve is indicated in functional notation by writing the equation in the form $f(x, y, \alpha) = 0$. The quantity α, which is constant for the same curve but different for different curves, is called the *parameter* of the family. The equations of two neighboring curves are then written

$$f(x, y, \alpha) = 0, \; f(x, y, \alpha + h) = 0,$$

in which h is a small increment of α. These curves can be brought as near to coincidence as desired by diminishing h.

118. Envelope of a family of curves. A point of intersection of two neighboring curves of the family tends toward a limiting position as the curves approach coincidence. The locus of such limiting points of intersection is called the *envelope* of the family.

Let $$f(x, y, \alpha) = 0, \qquad f(x, y, \alpha + h) = 0 \tag{1}$$

be two curves of the family. By the theorem of mean value (Art. 45)

$$f(x, y, \alpha + h) = f(x, y, \alpha) + h\frac{\partial f}{\partial \alpha}(x, y, \alpha + \theta h), \tag{2}$$

which, on account of equation (1), reduces to

$$\frac{\partial f}{\partial \alpha}(x, y, \alpha + \theta h) = 0.$$

Hence, it follows that in the limit, when $h \doteq 0$,

$$\frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0$$

is the equation of a curve passing through the limiting points of intersection of the curve $f(x, y, \alpha) = 0$ with its consecutive curve. This determines for any assigned value of α a definite limiting point of intersection on the corresponding member of the family. The locus of all such points is then to be obtained by eliminating the parameter α between the equations

$$f(x, y, \alpha) = 0, \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0.$$

The resulting equation is of the form $F(x, y) = 0$, and represents the fixed envelope of the family.

119. The envelope touches every curve of the family.

I. *Geometrical proof.* Let A, B, C be three consecutive curves of the family; let A, B intersect in P; B, C intersect in Q. When P, Q approach coincidence, PQ will be the direction of the tangent to the envelope at P; but since P, Q are two points on B that approach coincidence, hence PQ is also the direction of the tangent to B at P, and accordingly B and the envelope have a common tangent at P. Similarly for every curve of the family.

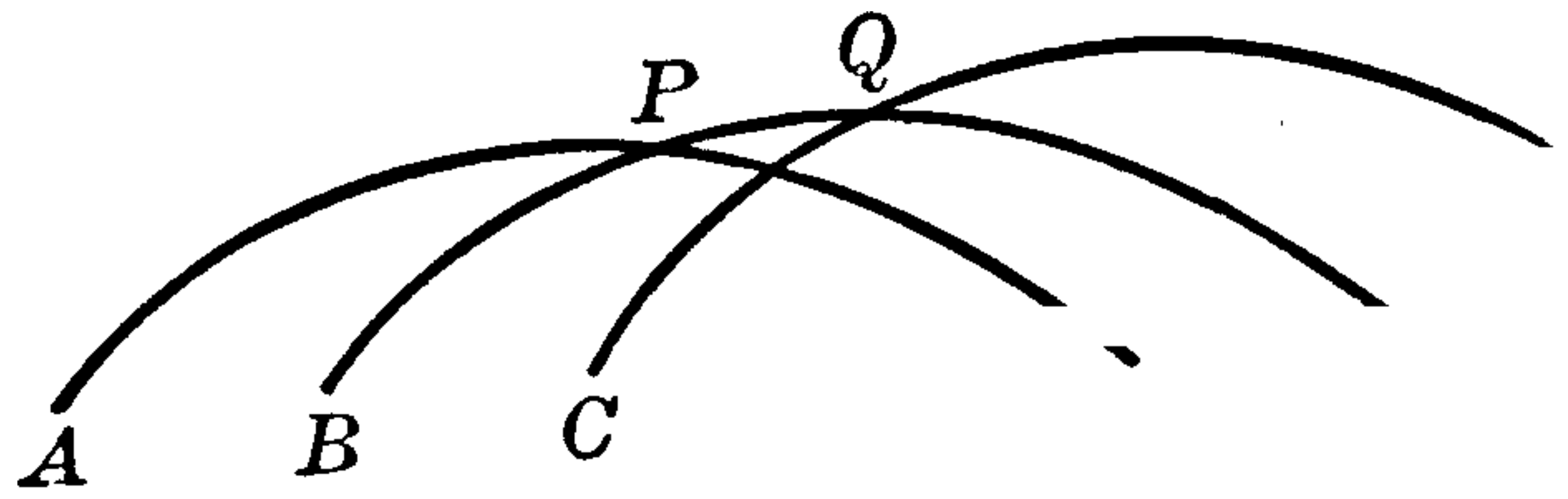


FIG. 58.

II. *More rigorous analytical proof.* Let $\frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0$ be solved for α, in the form $\alpha = \phi(x, y)$. Then the equation of the envelope is

$$f(x, y, \phi(x, y)) = 0.$$

Equating the total x-derivative to zero,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right) = 0;$$

but $\frac{\partial f}{\partial \phi}=\frac{\partial f}{\partial \alpha}=0$, hence the slope of the tangent to the envelope at the point (x, y) is given by

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0.$$

But this equation defines the direction of the tangent to the curve $f(x, y, \alpha)=0$ at the same point, and therefore a limiting point of intersection on any member of the family is a point of contact of this curve with the envelope.

Ex. Find the envelope of the family of lines

$$y = mx + \frac{p}{m}, \tag{1}$$

obtained by varying m.

Differentiate (1) as to m,

$$0 = x - \frac{p}{m^2}. \tag{2}$$

To eliminate m, multiply (2) by m and square; square (1) and subtract the first from the second. The envelope is found to be the parabola

$$y^2 = 4px.$$

120. Envelope of normals of a given curve. The evolute (Art. 110) was defined as the locus of the centers of curvature. The center of curvature was shown to be the point of intersection of consecutive normals (Art. 111), whence by Art. 118 the envelope of the normals is the evolute.

Ex. Find the envelope of the normals to the parabola $y^2 = 4px$.

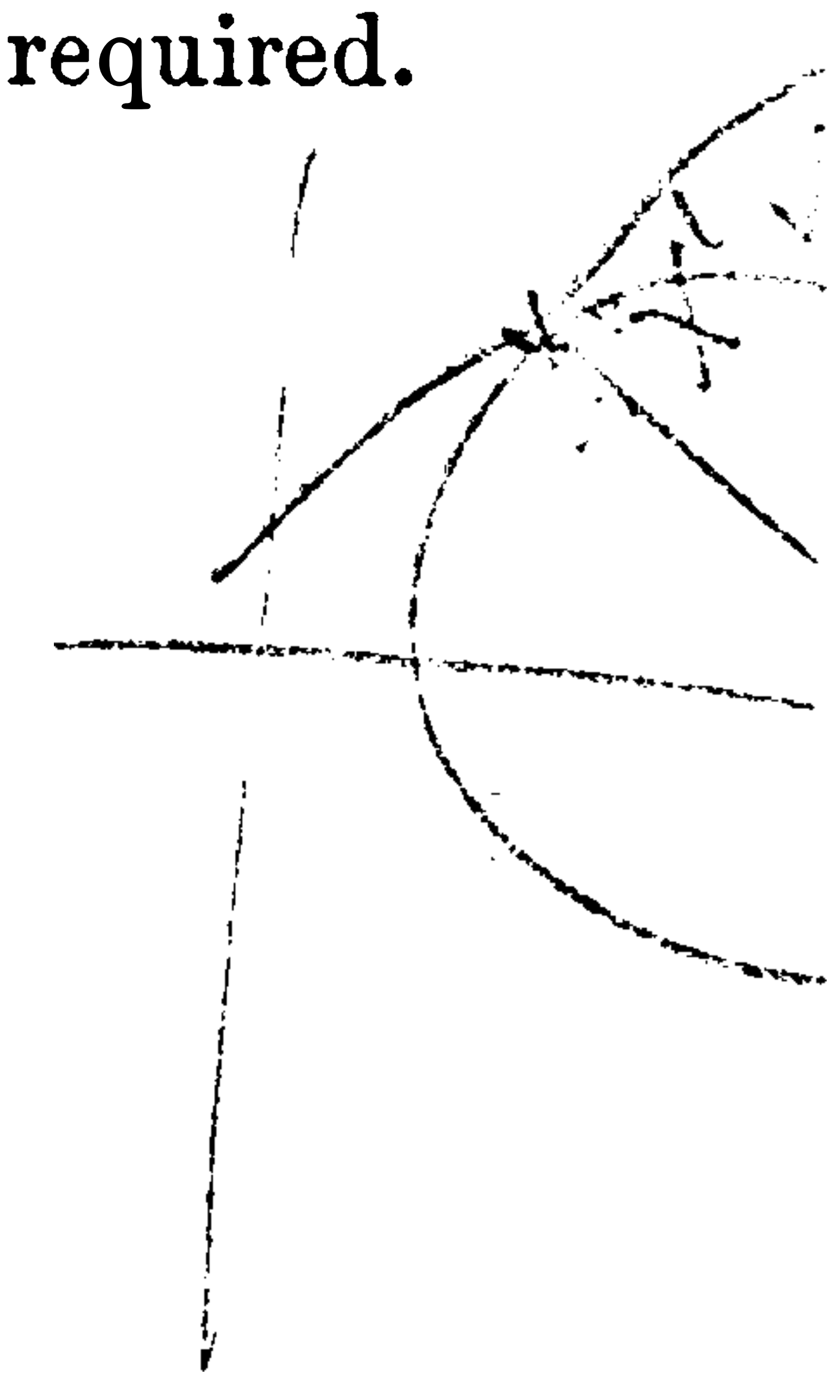
The equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{2p}(x - x_1),$$

or, eliminating x_1 by means of the equation $y_1^2 = 4px_1$,

$$y - y_1 = \frac{y_1^3}{8p^2} - \frac{xy_1}{2p}. \tag{1}$$

The envelope of this line, when y_1 takes all values, is required.



Differentiating as to y_1,

$$-1 = \frac{3y_1^2}{8p^2} - \frac{x}{2p},$$

$$y_1^2 = \frac{4}{3}p(x - 2p).$$

Substituting this value for y_1 in (1), the result,

$$27py^2 = 4(x - 2p)^3,$$

is the equation of the required evolute.

121. Two parameters, one equation of condition. In many cases a family of curves may have two parameters which are connected by an equation. For instance, the equation of the normal to a given curve contains two parameters x_1, y_1 which are connected by the equation of the curve. In such cases one parameter may be eliminated by means of the given relation, and the other treated as before.

When the elimination is difficult to perform, both equations may be differentiated as to one of the parameters α, regarding the other parameter β as a function of α. This gives four equations from which α, β and $\frac{d\beta}{d\alpha}$ may be eliminated, the resulting equation being that of the desired envelope.

Ex. **1.** Find the envelope of the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the sum of its intercepts remaining constant.

The two equations are

$$\frac{x}{a} + \frac{y}{b} = 1,$$

$$a + b = c.$$

Differentiate both equations as to a;

$$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0,$$

$$1+\frac{db}{da}=0.$$

Eliminate $$\frac{db}{da}.$$

Then $\frac{x}{a^2}=\frac{y}{b^2}$, which reduces to

$$\frac{\frac{x}{a}}{a}=\frac{\frac{y}{b}}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}; \text{ whence } a=\sqrt{cx},\ b=\sqrt{cy}.$$

Therefore $$\sqrt{x}+\sqrt{y}=\sqrt{c}$$

is the equation of the desired envelope. [Compare Ex. p. 131.]

Ex. **2**. Find the envelope of the family of coaxial ellipses having a constant area.

Here $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

$$ab=k^2.$$

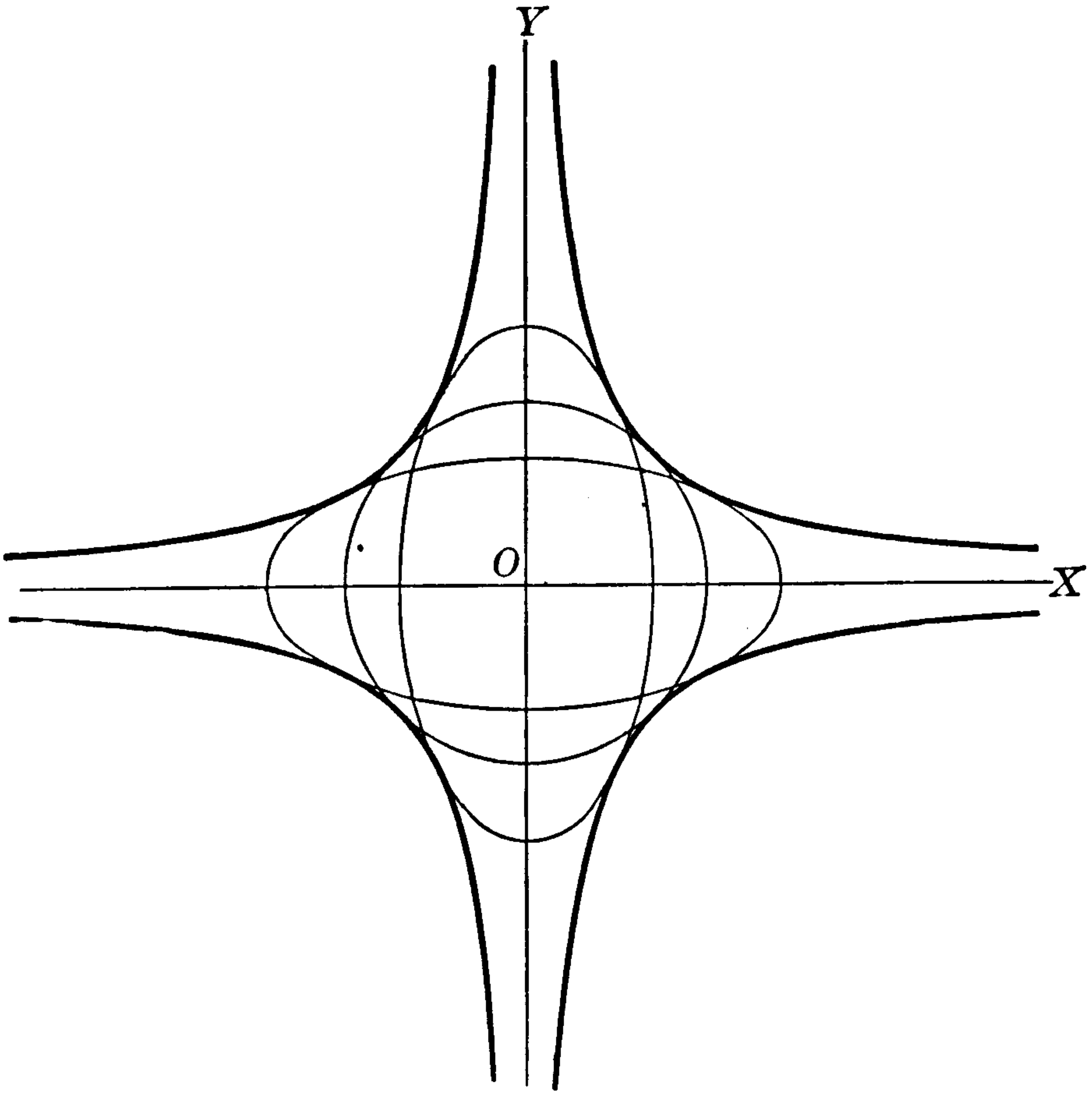


FIG. 59.

For symmetry, regard a and b as functions of a single parameter t.

Then
$$\frac{x^2}{a^3}da+\frac{y^2}{b^3}db=0,$$

$$bda+adb=0;$$

hence
$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{1}{2},$$

$$a=\pm x\sqrt{2},\ b=\pm y\sqrt{2},$$

and the envelope is the pair of rectangular hyperbolas $xy=\pm\frac{1}{2}k^2$.

NOTE. A family of curves may have no envelope; *i.e.*, consecutive curves may not intersect; *e.g.*, the family of concentric circles $x^2+y^2=r^2$, obtained by giving r all possible values.

If every curve of a family has a node, and the node has different positions for different curves of the family, the envelope will be composed of two (or more) curves, one of which is the locus of the node.

Ex. Find the envelope of the system

$$f\equiv(y-\lambda)^2+x^4-x^2=0,$$

in which λ is a varying parameter.

Here $\frac{\partial f}{\partial\lambda}=-2(y-\lambda)=0$; by combining with $f=0$ to eliminate λ, we obtain

$$x^2=0,\ x-1=0,\ x+1=0.$$

From Art. 114 it is seen that

$$x=0,\ y=\lambda$$

is a node on f; moreover, the various curves of the family are obtained by moving any one of them parallel to the y-axis. The lines $x-1=0$, $x+1=0$ form the proper envelope, and $x=0$ is the locus of the node.

EXERCISES ON CHAPTER XVI

1. Find the envelope of the line $x\cos\alpha+y\sin\alpha=p$, when α is a parameter.

2. A straight line of fixed length a moves with its extremities in two rectangular axes. Find its envelope.

✓ **3.** Ellipses are described with common centers and axes, and having the sum of the semi-axes equal to c. Find their envelope.

✓ **4.** Find the envelope of the straight lines having the product of their intercepts on the coördinate axes equal to k^2.

✓ **5.** Find the envelope of the lines $y - \beta = m(x - \alpha) + r\sqrt{1 + m^2}$, m being a variable parameter.

6. A circle moves with its center on a parabola whose equation is $y^2 = 4\,ax$, and passes through the vertex of the parabola. Find its envelope.

7. Find the envelope of a perpendicular to the normal to the parabola $y^2 = 4\,ax$, drawn through the intersection of the normal with the x-axis.

8. Show that the curves defined by the equations

$$\frac{\alpha}{x} + \frac{\beta}{y} = 1, \quad \alpha + \beta = c,$$

in which α and β are parameters, all pass through four fixed points; find them.

9. In the "nodal family" $(y - 2\,a)^2 = (x - a)^2 + 8\,x^3 - y^3$, show that the usual process gives for envelope a composite locus, made up of the "node-locus" (a line) and the envelope proper (an ellipse).

Mar 21.

INTEGRAL CALCULUS

CHAPTER I

GENERAL PRINCIPLES OF INTEGRATION

122. The fundamental problem. The fundamental problem of the Differential Calculus, as explained in the preceding pages, is this:

Given a function $f(x)$, of an independent variable x, to determine its derivative $f'(x)$.

It is now proposed to consider the inverse problem, viz.:

Given any function $f'(x)$, to determine the function $f(x)$ having $f'(x)$ for its derivative.

The study of this inverse problem is one of the objects of the Integral Calculus.

The given function $f'(x)$ is called the *integrand*, the function $f(x)$ which is to be found is called the *integral*, and the process gone through in order to obtain the unknown function $f(x)$ is called *integration*.

The operation and result of differentiation are symbolized by the formula

$$\frac{d}{dx}f(x) = f'(x), \tag{1}$$

or, written in the notation of differentials,

$$df(x) = f'(x)\,dx. \tag{2}$$

The operation of integration is indicated by prefixing the symbol $\int$ to the function, or differential, whose integral it is required to find. Accordingly, the formula of integration is written thus:

$$f(x) = \int f'(x)\,dx.$$

Following long established usage, the differential, rather than the derivative, of the unknown function $f(x)$ is written under the sign of integration. One of the advantages of so doing is that the variable, with respect to which the integration is performed, is explicitly mentioned. This is, of course, not necessary when only one variable is involved, but is essential when several variables enter into the integrand, or a change of variable is made during the process of integration.

123. Integration by inspection. The most obvious aid to the problem of integration is a knowledge of the rules and results of differentiation. It frequently happens that the required function $f(x)$ can be determined at once by recollecting the result obtained in some previous differentiation.

For example, suppose it to be required to find

$$\int \cos x\,dx.$$

It will be recalled that $\cos x\,dx$ is the differential of $\sin x$, and thus the answer to the proposed integration is directly obtained. That is,

$$\int \cos x\,dx = \sin x.$$

Again, suppose it is required to integrate

$$\int x^n\,dx,$$

where n is any constant (except -1). This problem immediately suggests the formula for differentiating a variable affected by a constant exponent [(6), p. 49]. When this formula is written

$$dx^{n+1} = (n+1)x^n\,dx,$$

or, what is the same thing,

$$d\left(\frac{x^{n+1}}{n+1}\right) = x^n\,dx,$$

it becomes obvious that

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

An exception to this result occurs when n has the value -1. For in that case it is apparent from (8), p. 50, that

$$\int x^{-1}\,dx = \int\frac{dx}{x} = \log x.$$

The method indicated in the above illustration may be designated as the method of *integration by inspection*. This is in fact the only method of practical service available. The object of the various devices suggested in the subsequent pages is to transform the given integrand, or to separate it into simpler elements in such a way that the method of inspection can be applied.*

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* When all has been done that can be accomplished in this direction, it will be found that a large portion of the field is yet unexplored and unknown, and that many functions exist whose integrals cannot be found. By this we mean that such integrals cannot be expressed in terms of functions already known. To illustrate, let it be imagined that the integral calculus had been discovered before the logarithm function was known. It would then have been impossible to express the integral $\int\frac{dx}{x}$ in terms of known functions. This integral might in consequence have led to the discovery of the function $\log x$. An exactly analogous thing, in fact, has happened in the attempt to integrate other expressions, and many important and hitherto unknown functions have been discovered in this way which have greatly enriched the entire field of mathematics.

124. The fundamental formulas of integration. When the formulas of differentiation (1)–(25), pp. 49–50, are borne in mind, the method of inspection referred to in the preceding article leads at once to the following fundamental integrals. Upon these sooner or later every integration must be made to depend.

I. $$\int u^n\,du = \frac{u^{n+1}}{n+1}.$$

II. $$\int \frac{du}{u} = \log u.$$

III. $$\int a^u\,du = \frac{a^u}{\log a}.$$

IV. $$\int e^u\,du = e^u.$$

V. $$\int \cos u\,du = \sin u.$$

VI. $$\int \sin u\,du = -\cos u.$$

VII. $$\int \sec^2 u\,du = \tan u.$$

VIII. $$\int \operatorname{cosec}^2 u\,du = -\cot u.$$

IX. $$\int \sec u \tan u\,du = \sec u.$$

X. $$\int \operatorname{cosec} u \cot u\,du = -\operatorname{cosec} u.$$

XI. $$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u, \text{ or } -\cos^{-1} u.$$

XII. $$\int \frac{du}{1+u^2} = \tan^{-1} u, \text{ or } -\cot^{-1} u.$$

XIII. $$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u, \text{ or } -\operatorname{cosec}^{-1} u.$$

XIV. $$\int \frac{du}{\sqrt{2u-u^2}} = \operatorname{vers}^{-1} u.$$

125. Certain general principles. In applying the above formulas of integration certain principles which follow from the rules of differentiation should be borne in mind.

(*a*) *The integral of the sum of a finite number of functions is equal to the sum of the integrals of each function taken separately.*

This follows from Art. 16.

For example,

$$\int \frac{x^2 - 1}{x} dx = \int x\,dx - \int \frac{dx}{x} = \frac{x^2}{2} - \log x.$$

(*b*) *A constant factor may be removed from one side of the sign of integration to the other.*

For, since

$$d(c \cdot u) = c \cdot du,$$

it follows that

$$\int c\,du = c \int du = cu.$$

To illustrate, let it be required to integrate

$$\int 5\,x^2\,dx.$$

The numerical factor 5 is first placed outside the sign of integration, after which formula **I** is applied. Accordingly,

$$\int 5\,x^2\,dx = 5 \int x^2\,dx = 5 \cdot \frac{x^3}{3}.$$

Again, suppose the integral

$$\int \frac{x}{x^2 + 1} dx$$

is to be found. It is readily noticed that except for the constant factor 2 the numerator of the integrand is the exact derivative of the denominator, and formula **II** would be applicable. All that is required, then, in order to reduce the given integral to a known form, is to multiply inside the sign of integration by 2 and outside by $\frac{1}{2}$. This gives

$$\int \frac{x\,dx}{x^2+1} = \tfrac{1}{2}\int \frac{2\,x\,dx}{x^2+1} = \tfrac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \tfrac{1}{2}\log(x^2+1).$$

In this connection it must not be forgotten that *an expression containing the variable of integration cannot be removed from one side of the sign of integration to the other.*

(*c*) *An arbitrary constant may be added to the result of integration.*

For, the derivative of a constant is zero, and hence

$$du = d(u + c),$$

from which follows

$$\int du = \int d(u + c) = u + c.$$

This constant is called the *constant of integration.*

It will be seen from this that the result of integration is not unique, but that any number of functions (differing from each other, however, only by an additive constant) can be found which have the same given expression for derivative. [Compare Art. 16, Cor. 2.]

Thus, any one of the functions x^2-1, x^2+1, x^2+a^2, $(x-a)(x+a)$, etc., will serve as a solution of the problem of integrating $\int 2\,x\,dx$.

It often happens that different methods of integration lead to different results. All such differences, however, can occur only in the constant terms.

For example,

$$\int 3(x+1)^2\,dx = 3\int (x+1)^2 d(x+1) = (x+1)^3$$
$$= x^3 + 3\,x^2 + 3x + 1.$$

Integration of the terms separately gives

$$\int 3x^2\,dx + \int 6x\,dx + \int 3\,dx = x^3 + 3x^2 + 3x,$$

a result which agrees with the preceding except in the constant term.

Again, from formula **XII,**

$$\int \frac{dx}{x^2+1} = \tan^{-1}x, \text{ or } -\cot^{-1}x.$$

It does not follow from this that $\tan^{-1}x$ is equal to $-\cot^{-1}x$. But they can differ at most by an additive constant. In fact, it is known from trigonometry that

$$-\cot^{-1}x = \tan^{-1}x + k\pi + \frac{\pi}{2},$$

where k is any integer.

In a similar manner the different results in formulas **XI** and **XIII** can be explained.

EXERCISES

Integrate the following:

1. $\int \sqrt{x}\,dx \left(= \int x^{\frac{1}{2}}\,dx\right)$.
2. $\int x^a\,dx$.
3. $\int \frac{dx}{\sqrt[3]{x}}$.
4. $\int \frac{mx^{m-1}}{\sqrt{x}}\,dx$.
5. $\int (a^{\frac{1}{3}} - x^{\frac{1}{3}})^3\,dx$.
6. $\int \frac{5x^3 - 3x + 1}{x^4}\,dx$.
7. $\int x(x^2 + a^2)^2\,dx$.
8. $\int (ax + b)^n\,dx$.
9. $\int \frac{dx}{x+a}$. rule II
10. $\int \frac{(a-x)\,dx}{2ax - x^2}$.
11. $\int \frac{\sec^2 x\,dx}{\tan x}$.
12. $\int \frac{\sin x\,dx}{1 + \cos x}$.
13. $\int \frac{dx}{x\log x} \left(= \int \frac{\frac{dx}{x}}{\log x}\right)$.
14. $\int \frac{5x^2\,dx}{x^3 + 1}$.

15. $\int \tan x\,dx\left(=-\int\frac{-\sin x\,dx}{\cos x}\right)$.

16. $\int \cot x\,dx$.

17. $\int e^{ax}\,dx$.

18. $\int e^{x^2}x\,dx$.

19. $\int (a+b)^{m+nx}\,dx$.

20. $\int \cos^2\tfrac{1}{2}x\,dx\left(=\int\frac{1+\cos x}{2}dx\right)$.

21. $\int \sin\,(m+n)x\,dx$.

22. $\int \sin x^2\cdot x\,dx$.

23. $\int \cos^3 x\,dx\left(=\int\cos x\,(1-\sin^2 x)\,dx\right)$.

24. $\int \sin^3 x\,dx$.

25. $\int \sin^2 x\,dx$.

26. $\int \tan^2 x\,dx\left[=\int(\sec^2 x-1)\,dx\right]$.

27. $\int \tan^2 x\sec^2 x\,dx$.

28. $\int \operatorname{cosec}^2(ax+b)\,dx$.

29. $\int \sqrt{\cot x}\operatorname{cosec}^2 x\,dx$.

30. $\int\frac{dx}{\sin x\cos x}\left(=\int\frac{\sec^2 x\,dx}{\tan x}\right)$.

31. $\int \sec^3 u\tan u\,du$.

32. $\int \cot u\,du\left(=\int\frac{\operatorname{cosec} u\cot u\,du}{\operatorname{cosec} u}\right)$.

33. $\int\frac{\tan u\,du}{\sec u}$.

34. $\int\frac{dx}{\sqrt{a^2-x^2}}\left(=\int\frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)$.

35. $\int\frac{dx}{\sqrt{1-4x^2}}$.

36. $\int\frac{du}{a^2+u^2}$.

37. $\int\frac{du}{a^2+b^2u^2}$.

38. $\int\frac{dx}{5+x^2-2x}\left(=\int\frac{dx}{(x-1)^2+4}\right)$

39. $\int\frac{dx}{(x-1)\sqrt{x^2-2x}}$.

40. $\int\frac{dx}{x\sqrt{x^2-a^2}}$.

126. Integration by parts. If u and v are functions of x, the rule for differentiating a product gives the formula

$$d(uv) = v\,du + u\,dv,$$

whence, by integrating and transposing terms,

$$\int u\,dv = uv - \int v\,du.$$

This formula affords a most valuable method of integration, known as integration by parts. By its use a given integral is made to depend on another integral, which in many important cases is of simpler form and more readily integrable than the original one.

Ex. 1. $$\int \log x\,dx.$$

Assume $u = \log x, \quad dv = dx.$

Then $du = \dfrac{dx}{x}, \qquad v = x.$

By substituting in the formula for integration by parts,

$$\begin{aligned}\int \log x\,dx &= x\log x - \int dx \\ &= x\log x - x = x(\log x - 1) \\ &= x(\log x - \log e) = x\log\frac{x}{e}.\end{aligned}$$

Ex. 2. $$\int xe^x\,dx.$$

Assume $u = x, \quad dv = e^x dx.$

Then $du = dx, \quad v = e^x,$

and $\displaystyle\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1).$

Suppose that a different choice had been made for u and dv in the present problem, say

$$u = e^x, \quad dv = x\,dx.$$

From this would follow

$$du = e^x dx, \quad v = \frac{x^2}{2},$$

and $$\int x e^x dx = \tfrac{1}{2} x^2 e^x - \int \frac{x^2}{2} e^x dx.$$

It will be observed that the new integral $\int \frac{x^2}{2} e^x dx$ is less simple in form than the original one, and hence the present choice of u and dv is not a fortunate one.

No general rule can be laid down for the selection of u and dv. Several trials may be necessary before a suitable one can be found.

It is to be remarked, however, that as far as possible dv should be chosen in such a way that its integral may be as simple as possible, while u should be so chosen that in differentiating it a material simplification is brought about. Thus in Ex. 1, by taking $u = \log x$, the transcendental function is made to disappear by differentiation. In Ex. 2, the presence of either x or e^x prevents direct integration. The first factor x can be removed by differentiation, and thus the choice $u = x$ is naturally suggested.

Ex. **3**. $$\int x^2 a^x dx.$$

From the preceding remark it is evident that the only choice which will simplify the integral is

$$u = x^2, \qquad dv = a^x dx.$$

Hence $$du = 2\,x\,dx, \quad v = \frac{a^x}{\log a},$$

and $$\int x^2 a^x dx = \frac{x^2 a^x}{\log a} - \frac{2}{\log a} \int x a^x dx.$$

Apply the same method to the new integral, assuming

$$u = x, \quad dv = a^x dx,$$

whence $$du = dx, \quad v = \frac{a^x}{\log a},$$

and $$\int x a^x dx = \frac{x a^x}{\log a} - \frac{1}{\log a} \int a^x dx$$

$$= \frac{x a^x}{\log a} - \frac{a^x}{(\log a)^2}.$$

By substituting in the preceding formula,

$$\int x^2 a^x dx = \frac{a^x}{\log a}\left[x^2 - \frac{2\,x}{\log a} + \frac{2}{(\log a)^2}\right].$$

EXERCISES

1. $\int \sin^{-1} x\, dx.$

2. $\int e^x \tan^{-1}(e^x)\, dx.$

3. $\int x^2 \cos x\, dx.$

4. $\int x^n \log x\, dx.$

5. $\int x^2 \tan^{-1} x\, dx.$

6. $\int \sec x \tan x \log \cos x\, dx.$

7. $\int x \cot^{-1} x\, dx.$

8. $\int x \sin 3\, x\, dx.$

9. $\int e^x \cos x\, dx.$

10. $\int e^x \sin x\, dx.$

11. $\int \cos x \cos 2\, x\, dx.$

Where

127. Integration by substitution. It is often necessary to simplify a given differential $f'(x)dx$ by the introduction of a new variable before integration can be effected. Except for certain special classes of differentials (see, for example, Arts. 138, 139) no general rule can be laid down for the guidance of the student in the use of this method, but some aid may be derived from the hints contained in the problems which follow.

Ex. 1. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}}.$

Introduce a new variable z by means of the substitution $a^2 - x^2 = z$. Differentiate and divide by -2, whence $x\, dx = -\frac{dz}{2}$. Accordingly

$$\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = -\tfrac{1}{2} \int \frac{dz}{\sqrt{z}} = -\tfrac{1}{2} \int z^{-\frac{1}{2}} dz = -z^{\frac{1}{2}} = -\sqrt{a^2 - x^2}.$$

The details required in carrying out this substitution are so simple that they can be omitted and the solution of the problem will then take the following form:

$$\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = \int (a^2 - x^2)^{-\frac{1}{2}} x\, dx = -\tfrac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} (-2\, x\, dx) = -(a^2 - x^2)^{\frac{1}{2}}.$$

In this series of steps the last integral is obtained by multiplying inside the sign of integration by -2 and outside by $-\frac{1}{2}$, the object being to make the second factor the differential of $a^2 - x^2$. Thinking of the latter as a new variable, the integrand contains this variable affected by an exponent $(-\frac{1}{2})$ and multiplied by the differential of the variable, in which case formula **I** can be applied.

Ex. 2. $\int \frac{\log x}{x} dx.$

Assume $$\log x = z.$$

Then $$\frac{dx}{x} = dz,$$

and $$\int \frac{\log x}{x} dx = \int z\, dz = \frac{z^2}{2} = \frac{(\log x)^2}{2}.$$

Here again it is not necessary to write out the details of the substitution, as it is easy to think of $\log x$ as a new independent variable and to perform the integration with respect to that. It is then readily seen that the expression to be integrated consists of the variable $\log x$ multiplied by its differential $\frac{dx}{x}$, and that the integration is accordingly reduced to an immediate application of the first formula of integration. Thus

$$\int \log x \cdot d(\log x) = \frac{(\log x)^2}{2}.$$

Ex. 3. $\int e^{\tan^{-1} x} \frac{dx}{1+x^2}.$

Think of $\tan^{-1} x$ as a new variable and apply formula **IV**. Thus

$$\int e^{\tan^{-1} x} \frac{dx}{1+x^2} = \int e^{\tan^{-1} x} d(\tan^{-1} x) = e^{\tan^{-1} x}.$$

Ex. 4. $\int \frac{\sin^{-1} x\, dx}{\sqrt{1-x^2}}.$

Think of $\sin^{-1} x$ as a new variable and $\frac{dx}{\sqrt{1-x^2}}$ as the differential of that variable. Apply formula **I**.

Ex. 5. $\int (x^2 + 2x + 3)(x+1) dx.$

Multiply and divide by 2. The integral then takes the form

$$\frac{1}{2}\int (x^2 + 2x + 3) \cdot (2x + 2) dx.$$

Observing that $(2x + 2)dx$ is the differential of $x^2 + 2x + 3$, and thinking of the latter expression as a new variable, it is seen that formula **I** is directly applicable, leading to the result

$$\frac{1}{4}(x^2 + 2x + 3)^2.$$

Ex. 6. $\int \log\cos(x^2+1)\sin(x^2+1)\cdot x\,dx.$

Make the substitution

$$x^2+1=z.$$

The given integral takes the form

$$\tfrac{1}{2}\int \log\cos z\,\sin z\,dz.$$

Make a second change of variable,

$$\cos z = y.$$

Then
$$\sin z\,dz = -\,dy.$$

The transformed integral is

$$-\tfrac{1}{2}\int \log y\,dy,$$

to which the result of Ex. 1, Art. 126, can be at once applied.

It will be observed that two substitutions which naturally suggest themselves from the form of the integrand are made in succession. The two together are obviously equivalent to the one transformation,

$$\cos(x^2+1)=y.$$

Ex. 7. $\int \frac{dx}{\sqrt{a^2-x^2}}.$

Either put $x = az$, or else divide numerator and denominator by a, and write in the form

$$\int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.$$

Regarding $\frac{x}{a}$ as a new variable, this comes under **XI** and gives the result

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$$
$$= -\cos^{-1}\frac{x}{a}+C'.$$

In a similar manner treat Exs. 8–10.

Ex. 8. $\int \frac{dx}{x^2+a^2}.$

Ex. 9. $\int \frac{dx}{x\sqrt{x^2-a^2}}.$

Try also the substitution $x = \frac{1}{z}.$

Ex. 10. $\int \frac{dx}{\sqrt{2ax - x^2}}$.

Try also the substitution $x - a = z$.

Ex. 11. $\int \frac{dx}{\sqrt{x^2 \pm a^2}}$.

Make the transformation

$$x + \sqrt{x^2 \pm a^2} = z.$$

From this follows, by differentiation,

$$\left(1 + \frac{x}{\sqrt{x^2 \pm a^2}}\right) dx = dz;$$

that is,
$$(\sqrt{x^2 \pm a^2} + x) \frac{dx}{\sqrt{x^2 \pm a^2}} = dz,$$

or,
$$\frac{dx}{\sqrt{x^2 \pm a^2}} = \frac{dz}{\sqrt{x^2 \pm a^2} + x} = \frac{dz}{z}.$$

Ex. 12. $\int \frac{dx}{x^2 - a^2}$.

Assume
$$\frac{x - a}{x + a} = z; \quad \text{that is, } x = a\frac{1 + z}{1 - z}.$$

The reasons for the choice of substitution made in this and the preceding example will be made clear in Arts. 133 and 139.

Ex. 13. $\int \operatorname{cosec} x \, dx$.

Multiply and divide by $\operatorname{cosec} x - \cot x$. It will be readily seen that the integral then takes the form $\int \frac{dz}{z}$.

Another method would be to use the trigonometric formula

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2},$$

whence
$$\int \operatorname{cosec} x \, dx = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2} \, d\left(\frac{x}{2}\right)}{\tan \frac{x}{2}}.$$

Ex. 14. $\int \sec x \, dx$.

Put $x = z - \frac{\pi}{2}$, and use Ex. 13.

Solve the problem also by means of substitutions similar to those used in the preceding example.

there

Ex. 15. $$\int\frac{dx}{ax^2+bx+c}=\int\frac{4a\,dx}{4a^2x^2+4abx+4ac}$$

$$=2\int\frac{dz}{z^2+4ac-b^2} \qquad (z=2ax+b)$$

$$=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}},\ \text{if } 4ac-b^2>0.$$

$$=\frac{1}{\sqrt{b^2-4ac}}\log\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}},\ \text{if } 4ac-b^2<0.$$

Ex. 16. $$\int\frac{dx}{2x^2+2x+3}=\int\frac{2\,dx}{4x^2+4x+6}=\int\frac{d(2x)}{(2x+1)^2+5}.$$

In this form it is easy to integrate by taking $2x+1$ as a new variable.

Ex. 17. $$\int\frac{dx}{3x^2-2x+5}.$$

Ex. 18. $$\int\frac{4\,dx}{8+4x-4x^2}.$$

Ex. 19. $$\int\frac{dx}{\sqrt{-9x^2+30x-24}}.$$

Ex. 20. $$\int(3x-2)\cos(3x-2)\,dx.$$

Ex. 21. $$\int\frac{a\,dx}{x\sqrt{a^2+bx}}.$$

Substitute $\sqrt{a^2+bx}=z$, and use Ex. 12.

128. Additional standard forms. The integrals in Exs. 7–14 of the preceding article, and in Exs. 15–16 of Art. 125, are of such frequent occurrence that it is desirable to collect the results of integration into an additional list of standard forms.

XV. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$, or $-\cos^{-1}\frac{u}{a}$.

XVI. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log(u + \sqrt{u^2 \pm a^2})$.

XVII. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$, or $-\frac{1}{a}\cot^{-1}\frac{u}{a}$.

XVIII. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{u-a}{u+a}$.

XIX. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}$, or $-\frac{1}{a}\operatorname{cosec}^{-1}\frac{u}{a}$.

XX. $\int \frac{du}{\sqrt{2au - u^2}} = \operatorname{vers}^{-1}\frac{u}{a}$.

XXI. $\int \tan u\, du = -\log\cos u = \log\sec u$.

XXII. $\int \cot u\, du = \log\sin u$.

XXIII. $\int \sec u\, du = \log(\sec u + \tan u) = \log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right)$.

XXIV. $\int \operatorname{cosec} u\, du = \log(\operatorname{cosec} u - \cot u) = \log\tan\frac{u}{2}$.

129. Integrals of the form

$$\int \frac{(Ax + B)dx}{\sqrt{ax^2 + bx + c}}.$$

Integrals of this form are of such frequent occurrence as to deserve special mention. The integration is readily effected by the substitution of a new variable which reduces the radical to a simpler form. Two cases are to be considered according as a is positive or negative.

CASE I. *a positive.* In this case by dividing out the coefficient of x^2 the radical may be written

$$\sqrt{a}\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}} = \sqrt{a}\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}}.$$

The given integral then takes the form

$$\frac{1}{\sqrt{a}}\int\frac{(Ax+B)dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}=\frac{1}{\sqrt{a}}\int\frac{\left(Az+B-\frac{bA}{2a}\right)dz}{\sqrt{z^2+\frac{4ac-b^2}{4a^2}}}\left(z=x+\frac{b}{2a}\right)$$

$$=\frac{A}{\sqrt{a}}\int\frac{z\,dz}{\sqrt{z^2+\frac{4ac-b^2}{4a^2}}}+\frac{2aB-bA}{2a\sqrt{a}}\int\frac{dz}{\sqrt{z^2+\frac{4ac-b^2}{4a^2}}}$$

$$=\frac{A}{\sqrt{a}}\sqrt{z^2+\frac{4ac-b^2}{4a^2}}+\frac{2aB-bA}{2a\sqrt{a}}\log\left(z+\sqrt{z^2+\frac{4ac-b^2}{4a^2}}\right)$$

$$=\frac{A}{a}\sqrt{ax^2+bx+c}+\frac{2aB-bA}{2a\sqrt{a}}\log\left(x+\frac{b}{2a}+\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\right).$$

CASE II. *a negative.* When a is negative, by dividing out the positive number $-a$ the radical becomes

$$\sqrt{-a}\sqrt{-x^2-\frac{b}{a}x-\frac{c}{a}}=\sqrt{-a}\sqrt{\frac{b^2-4ac}{4a^2}-\left(x+\frac{b}{2a}\right)^2},$$

and in consequence the integral takes the form

$$\frac{1}{\sqrt{-a}}\int\frac{(Ax+B)dx}{\sqrt{\frac{b^2-4ac}{4a^2}-\left(x+\frac{b}{2a}\right)^2}}$$

$$=\frac{1}{\sqrt{-a}}\int\frac{\left(Az+B-\frac{bA}{2a}\right)dz}{\sqrt{\frac{b^2-4ac}{4a^2}-z^2}}\qquad\left(z=x+\frac{b}{2a}\right)$$

$$=\frac{A}{\sqrt{-a}}\int\frac{z\,dz}{\sqrt{\frac{b^2-4ac}{4a^2}-z^2}}+\frac{2aB-bA}{2a\sqrt{-a}}\int\frac{dz}{\sqrt{\frac{b^2-4ac}{4a^2}-z^2}}$$

$$=\frac{-A}{\sqrt{-a}}\sqrt{\frac{b^2-4ac}{4a^2}-z^2}+\frac{2aB-bA}{2a\sqrt{-a}}\sin^{-1}\frac{2az}{\sqrt{b^2-4ac}}$$

$$=\frac{A}{a}\sqrt{ax^2+bx+c}+\frac{2aB-bA}{2a\sqrt{-a}}\sin^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}.$$

Ex. 1. $\displaystyle\int\frac{(2x+4)\,dx}{\sqrt{3x^2+3x+2}}$.

On dividing numerator and denominator by $\sqrt{3}$ the integral reduces to

$$\int\frac{\left(\frac{2}{\sqrt{3}}x+\frac{4}{\sqrt{3}}\right)dx}{\sqrt{(x+\frac{1}{2})^2+\frac{5}{12}}},$$

which by means of the substitution $x+\frac{1}{2}=z$ can be written

$$\int\frac{\left(\frac{2}{\sqrt{3}}z+\sqrt{3}\right)dz}{\sqrt{z^2+\frac{5}{12}}}=\frac{1}{\sqrt{3}}\int(z^2+\tfrac{5}{12})^{-\frac{1}{2}}(2z\,dz)+\sqrt{3}\int\frac{dz}{\sqrt{z^2+\frac{5}{12}}}$$

$$=\frac{1}{\sqrt{3}}\frac{(z^2+\frac{5}{12})^{\frac{1}{2}}}{\frac{1}{2}}+\sqrt{3}\log\left(z+\sqrt{z^2+\tfrac{5}{12}}\right)$$

$$=\frac{2}{\sqrt{3}}\sqrt{x^2+x+\tfrac{2}{3}}+\sqrt{3}\log\left(x+\tfrac{1}{2}+\sqrt{x^2+x+\tfrac{2}{3}}\right).$$

Ex. 2. $\displaystyle\int\frac{(2x+1)\,dx}{\sqrt{-2x^2-3x-1}}$.

Divide numerator and denominator by $\sqrt{2}$. The integral becomes

$$\int\frac{\frac{2x+1}{\sqrt{2}}dx}{\sqrt{\frac{1}{16}-(x+\frac{3}{4})^2}}=\sqrt{2}\int\frac{z\,dz}{\sqrt{\frac{1}{16}-z^2}}-\frac{1}{2\sqrt{2}}\int\frac{dz}{\sqrt{\frac{1}{16}-z^2}}\qquad(z=x+\tfrac{3}{4})$$

$$=-\sqrt{2}\sqrt{\tfrac{1}{16}-z^2}-\frac{1}{2\sqrt{2}}\sin^{-1}4z$$

$$=-\sqrt{-2x^2-3x-1}-\frac{1}{2\sqrt{2}}\sin^{-1}(4x+3).$$

130. Integrals of the form $\displaystyle\int\frac{dx}{(Ax+B)\sqrt{ax^2+bx+c}}$.

Integrals of this type can be reduced to the form given in the preceding article by means of the reciprocal substitution

$$Ax+B=\frac{1}{z}.$$

From this follow the relations

$$A\,dx = -\frac{dz}{z^2},$$

and $$\sqrt{ax^2+bx+c} = \frac{1}{Az}\sqrt{\alpha z^2+\beta z+\gamma},$$

in which $$\alpha = aB^2 - bAB + cA^2,$$

$$\beta = -2\,aB + bA,$$

$$\gamma = a.$$

When these expressions are substituted in the given integral it reduces to

$$-\int\frac{dz}{\sqrt{\alpha z^2+\beta z+\gamma}},$$

which has the form discussed in Art. 129.

EXERCISES ON CHAPTER I

1. $\displaystyle\int\frac{x\,dx}{\sqrt{x^2+2\,x+2}}.$

2. $\displaystyle\int\frac{dx}{x\sqrt{5\,x^2-4\,x+1}}.$

3. $\displaystyle\int\frac{(2\,x-3)\,dx}{\sqrt{3\,x^2+x-2}}.$

4. $\displaystyle\int\frac{(4\,x+5)\,dx}{\sqrt{8+4\,x-4\,x^2}}.$

5. $\displaystyle\int\frac{x\,dx}{\sqrt{-x^2+2\,x+1}}.$

6. $\displaystyle\int\sqrt{\frac{1-x}{1+x}}\,dx.$

[Rationalize the numerator.]

Solve

7. $\displaystyle\int\sqrt{\frac{a-x}{x}}\,dx.$

8. $\displaystyle\int\frac{x^2+x+1}{(x+1)\sqrt{x^2+2\,x+3}}\,dx.$

[Divide the numerator by $x+1$.]

9. $\displaystyle\int\frac{\sqrt{1+x+x^2}}{1+x}\,dx.$

10. $\displaystyle\int\frac{(x^2-1)\,dx}{x\sqrt{x^4+x^2+1}}.$

[Assume $x+\frac{1}{x}=z$.]

11. $\displaystyle\int e^{e^x}e^x\,dx.$

12. $\displaystyle\int\frac{5\,x^3\,dx}{4+x^8}.$ Solve

13. $\displaystyle\int\frac{(2+3\,x^2)\,dx}{6\,x^3+12\,x+5}.$

14. $\int \frac{dx}{\sqrt{1-e^{2x}}}.$

[Put $e^{-x} = z.$]

15. $\int \frac{dx}{\sqrt{4x^4+8x^2}}.$

16. $\int \frac{x\,dx}{\sqrt{1-x^4}}.$

17. $\int \frac{dx}{e^x - e^{-x}}.$

18. $\int x^4 \tan^{-1} x\,dx.$

three

19. $\int \frac{x^2-2x+3}{x^2+1}\,dx.$

20. $\int \frac{x^2\,dx}{a^x}.$

21. $\int \frac{d\theta}{\sin\theta+1}\left[=\int \frac{1-\sin\theta}{\cos^2\theta}d\theta\right].$

22. $\int \frac{d\theta}{1-\cos\theta}.$

23. $\int \frac{\tan x\,dx}{a+b\tan^2 x}.$

24. $\int \frac{dx}{1+\cot x}.$

$\left[\text{Put } 1+\cot x = \frac{1}{z}.\right]$

25. $\int \frac{x^3\,dx}{\sqrt{x^2-1}}. = \int x^2 . \frac{x\,dx}{\sqrt{x^2-1}}$

26. $\int \frac{x^{\frac{1}{3}}+2x^{\frac{1}{6}}}{x^{\frac{1}{6}}+1}\,dx.$

27. $\frac{1}{6}\int \frac{dx}{x\sqrt{1-\log x}}.$

28. $\int \frac{e^x\,dx}{e^x+e^{-x}}.$

29. $\int \frac{\cos x\,dx}{\sqrt{1+\cos^2 x - \sin x}}.$

30. $\int \frac{dx}{x(\log x)^2 + x}.$

31. $\int \left(\frac{\sec x}{a-b\tan x}\right)^2 dx = \int \frac{\sec^2 x\,dx}{(a-b\tan x)^2}$

32. $\int \frac{(x-a)\,dx}{\sqrt{a^4-a^2(x-a)^2-(x-a)^4}}.$

33. $\int \text{vers}^{-1}\frac{x}{a}\,\frac{dx}{\sqrt{x}}.$

34. $\int \frac{dx}{x^2\sqrt{3x^2+2x+1}}.$

$\left[\text{Substitute } x = \frac{1}{z}.\right]$

35. $\int \log\left(x+\sqrt{x^2-a^2}\right)\frac{dx}{x^3}.$

CHAPTER II

REDUCTION FORMULAS

131. In Arts. 129, 130 the integration of certain simple expressions containing an irrationality of the form $\sqrt{ax^2+bx+c}$ has been explained. As was shown in Art. 129, the radical can be reduced to the form $\sqrt{\pm x^2 \pm a^2}$ by a change of variable. It remains to show how the integration can be performed in such cases as, for example,

$$\int x^n\sqrt{\pm x^2 \pm a^2}\,dx, \qquad \int \frac{x^n dx}{\sqrt{\pm x^2 \pm a^2}},$$

n being any integer.

For this purpose it is convenient to consider a more general type of integral of which the preceding are special cases, viz.,

$$\int x^m(a+bx^n)^p dx, \tag{1}$$

in which m, n, p are any numbers whatever, integral or fractional, positive or negative.

It is to be remarked in the first place that n can, without loss of generality, be regarded as positive. For, if n were negative, say $n = -n'$, the integrand could be written

$$x^m\left(a+\frac{b}{x^{n'}}\right)^p = x^m\left(\frac{ax^{n'}+b}{x^{n'}}\right)^p = x^{m-pn'}(b+ax^{n'})^p.$$

This expression, which is of the same type as $x^m(a+bx^n)^p$, is such that the exponent of x inside the parenthesis is positive.

It will now be proved that *an integral of the type* (1) *can in general be reduced to one of the four integrals*

$$(a)\ A\int x^{m-n}(a+bx^n)^p dx, \qquad (b)\ A\int x^{m+n}(a+bx^n)^p dx,$$

$$(c)\ A\int x^m(a+bx^n)^{p-1} dx, \qquad (d)\ A\int x^m(a+bx^n)^{p+1} dx,$$

plus an algebraic term of the form

$$Bx^\lambda(a+bx^n)^\mu.$$

Here A, B, λ, μ are certain constants which will be determined presently.

Observe that in each of the four cases the integral to which (1) is reduced is of the same type as (1), but that certain changes have taken place in the exponents, viz.,

the exponent m of the monomial factor is increased or diminished by n,

or, the exponent p of the binomial is increased or diminished by unity.

The values of λ and μ are determined by the following rule:

Compare the exponents of the monomial factors in the given integral and in the integral to which it is to be reduced. Select the less of the two numbers and increase it by unity. The result is the value of λ. *In like manner, compare the exponents of the binomial factors in the two integrals, select the less, and increase by unity. This gives* μ.

Thus, if it is desired to reduce the given integral to

$$A\int x^{m-n}(a+bx^n)^p dx,$$

first write down the formula

$$\int x^m(a+bx^n)^p dx = A\int x^{m-n}(a+bx^n)^p dx + Bx^\lambda(a+bx^n)^\mu.$$

The exponents of the monomial factors in the two integrals are m and $m - n$ respectively, of which $m - n$ is the less. This, increased by unity, gives the value of λ; that is, $\lambda = m - n + 1$.

Again, the exponent of the binomial factor in each integral is the same, namely p, so that there is no choice as to which of the two is the less. Increase this number p by unity to obtain the value of μ. Hence $\mu = p + 1$.

The above formula may now be written

$$\int x^m(a+bx^n)^p dx$$
$$= A\int x^{m-n}(a+bx^n)^p dx + Bx^{m-n+1}(a+bx^n)^{p+1}. \quad (2)$$

In order to determine the values of the unknown constants A and B, simplify the equation by differentiating both members. After dividing by $x^{m-n}(a + bx^n)^p$ the resulting equation reduces to

$$x^n = A + Ba(m - n + 1) + Bb(m + np + 1)x^n.$$

By equating coefficients of like powers of x in both members, the values of A and B are found to be

$$A = -\frac{a(m - n + 1)}{b(m + np + 1)}, \quad B = \frac{1}{b(m + np + 1)}.$$

When these values are substituted in formula (2), it becomes

$$\int x^m(a + bx^n)^p dx$$
$$= -\frac{a(m-n+1)}{b(m+np+1)}\int x^{m-n}(a+bx^n)^p dx + \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(m+np+1)}. \quad [\text{A}]$$

Notice that the existence of formula (2) has been proved by showing that values can be found for A and B which make the two members of this equation identical.

There is one case, however, in which this reduction is impossible, viz., when

$$m + np + 1 = 0,$$

for in that case A and B become infinite. [See Ex. 4, p. 221.]

In a similar manner the three following formulæ may be derived:

$$\int x^m(a+bx^n)^p dx$$

$$= -\frac{b(m+n+np+1)}{a(m+1)}\int x^{m+n}(a+bx^n)^p dx + \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)}. \quad [\text{B}]$$

$$\int x^m(a+bx^n)^p dx$$

$$= \frac{anp}{m+np+1}\int x^m(a+bx^n)^{p-1}dx + \frac{x^{m+1}(a+bx^n)^p}{m+np+1}. \quad [\text{C}]$$

$$\int x^m(a+bx^n)^p dx$$

$$= \frac{m+n+np+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}dx - \frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)}. \quad [\text{D}]$$

The cases in which the above reductions are impossible are,

For formulæ [A] and [C], when $m + np + 1 = 0$;
for formula [B] , when $m + 1 = 0$;
for formula [D] , when $p + 1 = 0$.

Ex. 1. $\int x^3\sqrt{a^2 - x^2}\,dx$.

If the monomial factor were x instead of x^3, the integration could easily be effected by using formula **I**. Since in the present case $m = 3$, $n = 2$, formula [A], which diminishes m by n, will reduce the above integral to one that can be directly integrated.

Instead of substituting in [A], as might readily be done, it is best to apply to particular problems the same mode of procedure that was used in deriving the general formula. There are two advantages in this. First, it makes the student independent of the formulas, and second, when several reductions have to be made in the same problem, the work is generally shorter. [See Ex. 4.]

Accordingly assume

$$\int x^3(a^2-x^2)^{\frac{1}{2}}dx = A\int x(a^2-x^2)^{\frac{1}{2}}dx + Bx^2(a^2-x^2)^{\frac{3}{2}},$$

the values of λ and μ having been determined by the previously given rule.

Differentiate, and divide the resulting equation by $x(a^2-x^2)^{\frac{1}{2}}$. This gives

$$x^2 = A + B(2a^2-5x^2),$$

from which, by equating coefficients of like powers of x,

$$A = \frac{2a^2}{5},\quad B = -\tfrac{1}{5},$$

and hence,

$$\int x^3\sqrt{a^2-x^2}\,dx = \frac{2a^2}{5}\int(a^2-x^2)^{\frac{1}{2}}x\,dx - \tfrac{1}{5}x^2(a^2-x^2)^{\frac{3}{2}}$$

$$= -\tfrac{1}{15}(2a^2+3x^2)(a^2-x^2)^{\frac{3}{2}}.$$

Ex. **2.** $\int\sqrt{x^2-2x-3}\,dx.$

By following the suggestions of Art. 129, this integral can be reduced to the form

$$\int\sqrt{z^2-4}\,dz,$$

in which $z = x-1$.

Assume

$$\int(z^2-4)^{\frac{1}{2}}dz = A\int(z^2-4)^{-\frac{1}{2}}dz + Bz(z^2-4)^{\frac{1}{2}}.$$

In determining λ notice that $m=0$ in both integrals, so that $\lambda = 0+1 = 1$. Also, $\mu = -\frac{1}{2}+1 = \frac{1}{2}$.

Ex. **3.** $\int\sqrt{2ax-x^2}\,dx.$

The mode of procedure of Ex. 2 may be followed. Another method can also be used, as follows:

On writing in the form

$$\int x^{\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx,$$

and observing that

$$\text{vers}^{-1}\frac{x}{a} = \int\frac{dx}{\sqrt{2ax-x^2}} = \int x^{-\frac{1}{2}}(2a-x)^{-\frac{1}{2}}\,dx,$$

it will be seen that the integration may be effected in the present case by reducing each of the exponents m and p by unity. This is possible since $n=1$ and m can accordingly be diminished by 1. Hence assume

$$\int x^{\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx = A'\int x^{-\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx + B'x^{\frac{1}{2}}(2a-x)^{\frac{3}{2}}.$$

The exponent of the binomial in the new integral may be reduced in turn by assuming

$$\int x^{-\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx = A''\int x^{-\frac{1}{2}}(2a-x)^{-\frac{1}{2}}\,dx + B''x^{\frac{1}{2}}(2a-x)^{\frac{1}{2}}.$$

When this expression is substituted for the integral in the second member of the preceding equation, the result takes the form

$$\int\sqrt{2ax-x^2}\,dx = A\int\frac{dx}{\sqrt{2ax-x^2}} + Bx^{\frac{1}{2}}(2a-x)^{\frac{1}{2}} + Cx^{\frac{1}{2}}(2a-x)^{\frac{3}{2}},$$

in which A, B, C are written for brevity in the place of $A'A''$, $A'B''$, B' respectively. The values of A, B, C are calculated in the usual manner by differentiating, simplifying, and equating coefficients of like powers of x.

The method just given requires two reductions, and hence is less suitable than that employed in Ex. 2, which requires but one reduction.

The rule for determining the values of λ and μ may now be advantageously abbreviated. Let m, p be the exponents of the two factors in the given integral, and m', p' the corresponding exponents in the new integral. Of these two pairs, m, p and m', p', one of the numbers in the one pair is less than the corresponding number in the other pair. This fact will be expressed briefly by saying that the one pair is less than the other pair. With this understanding the preceding rule may be expressed as follows:

Select the less of the two pairs of exponents m, p and m', p'. Increase each number in the pair selected by unity. This gives the pair of exponents λ, μ.

Ex. 4. $\displaystyle\int\frac{x^4\,dx}{(x^2+a^2)^{\frac{3}{2}}}$.

Assume successively

$$\int x^4(x^2+a^2)^{-\frac{3}{2}}\,dx = A'\int x^4(x^2+a^2)^{-\frac{1}{2}}\,dx + B'x^5(x^2+a^2)^{-\frac{1}{2}},$$

$$\int x^4(x^2+a^2)^{-\frac{1}{2}}\,dx = A''\int x^2(x^2+a^2)^{-\frac{1}{2}}\,dx + B''x^3(x^2+a^2)^{\frac{1}{2}},$$

$$\int x^2(x^2+a^2)^{-\frac{1}{2}}\,dx = A'''\int (x^2+a^2)^{-\frac{1}{2}}\,dx + B'''x(x^2+a^2)^{\frac{1}{2}}.$$

These equations may be combined into the single formula

$$\int x^4(x^2+a^2)^{-\frac{3}{2}}\,dx = A\int (x^2+a^2)^{-\frac{1}{2}}\,dx + Bx\,(x^2+a^2)^{\frac{1}{2}}$$
$$+ Cx^3(x^2+a^2)^{\frac{1}{2}} + Dx^5(x^2+a^2)^{-\frac{1}{2}}.$$

The values of the coefficients are found to be

$$A = -\tfrac{3}{2}a^2,\quad B = \tfrac{3}{2},\quad C = -\frac{1}{a^2},\quad D = \frac{1}{a^2}.$$

Hence

$$\int x^4(x^2+a^2)^{-\frac{3}{2}}\,dx = \frac{x^3+3\,a^2x}{2\sqrt{x^2+a^2}} - \tfrac{3}{2}\,a^2\log\left(x+\sqrt{x^2+a^2}\right).$$

In this example three reductions were necessary; first, a reduction of type [*D*], second, and third, a reduction of type [*A*]. Can these reductions be taken in any order?

The different possible arrangements of the order in which these three reductions might succeed each other are

(1) [*A*], [*A*], [*D*]; (2) [*A*], [*D*], [*A*]; (3) [*D*], [*A*], [*A*],

of which number (3) was chosen in the solution of the problem. Of the other two arrangements, (2) can be used, but (1) cannot. For, after first applying [*A*] (which would be done in either case), the new integral is

$$\int x^2(a^2+x^2)^{-\frac{3}{2}}\,dx.$$

If [*A*] were now applied, it would be necessary to assume

$$\int x^2(a^2+x^2)^{-\frac{3}{2}}\,dx = A\int (a^2+x^2)^{-\frac{3}{2}} + Bx(a^2+x^2)^{-\frac{1}{2}}.$$

This equation, when differentiated and simplified, becomes

$$x^2 = A + Ba^2,$$

a relation which it is clearly impossible to reduce to an identity by equating coefficients of like powers of x, since there is no x^2 term in the right member to correspond with the one in the left member. It will be observed that this is the exceptional case mentioned on page 218, in which $m + np + 1 = 0$.

Ex. **5**. Show that the integral $\int \frac{(x^2-a^2)^{\frac{3}{2}}}{x^4}\,dx$ can be integrated by four reductions. Prove that these can be arranged in six different orders, and determine those which can be used.

Ex. 6. $\int \frac{dx}{(x^2+4)^2}$.

Ex. 7. $\int \frac{dx}{(x^2-x+1)^2}$.

Ex. 8. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$.

ok **Ex. 9.** $\int \sqrt{a^2-x^2}\,dx$.

ok **Ex. 10.** $\int \frac{dx}{x^2\sqrt{a^2-x^2}}$.

Ex. 11. $\int \frac{dx}{(x^2+a)^{\frac{5}{2}}}$. ✓

Ex. 12. $\int (a^2+x^2)^{\frac{3}{2}}\,dx$. ✓

ok **Ex. 13.** $\int \sqrt{x^2+a}\,dx$.

Ex. 14. $\int x\sqrt{2ax-x^2}\,dx$.

[illegible]

Ex. 15. $\int \frac{\sqrt{2ax-x^2}}{x^3}\,dx$. Rule B

Ex. 16. $\int \frac{dx}{\sqrt{x^6+x^{10}}}$. Rule B

Ex. 17. $\int \frac{dx}{(x^2+4x+3)^3}$. $(x+2)^2=$

Ex. 18. $\int \sqrt{1-2x-x^2}\,dx$. ✓

Ex. 19. Show that

$$\int \frac{dx}{(x^2+c)^n} = \frac{1}{2c(n-1)}\left[\frac{x}{(x^2+c)^{n-1}} + (2n-3)\int \frac{dx}{(x^2+c)^{n-1}}\right].$$

CHAPTER III

INTEGRATION OF RATIONAL FRACTIONS

132. Decomposition of rational fractions. The object of the present chapter is to show how to integrate fractions of the form

$$\frac{\phi(x)}{\psi(x)},$$

wherein $\phi(x)$ and $\psi(x)$ are polynomials in x.

The desired result is accomplished by the method of separating the given fraction into a sum of terms of a simpler kind, and integrating term by term.

If the degree of the numerator is equal to or greater than the degree of the denominator, the indicated division can be carried out until a remainder is obtained which is of lower degree than the denominator. Hence the fraction can be reduced to the form

$$\frac{\phi(x)}{\psi(x)} = ax^n + bx^{n-1} + \cdots + \frac{f(x)}{\psi(x)},$$

in which the degree of $f(x)$ is less than that of $\psi(x)$.

As to the integration of the remainder fraction $\frac{f(x)}{\psi(x)}$, it is to be remarked in the first place that the methods of the preceding articles are sufficient to effect the integration of such simple fractions as

$$\frac{A}{x-a}, \frac{A'}{(x-a)^2}, \cdots; \quad \frac{Mx+n}{x^2 \pm a^2}, \frac{M'x+N'}{(x^2 \pm a^2)^2}, \cdots; \quad \frac{Px+Q}{x^2+mx+n}, \cdots. \qquad (1)$$

Now the sum of several such fractions is a fraction of the kind under consideration, viz., one whose numerator is of lower degree than its denominator. The question naturally arises as to whether the converse is possible, that is: *can every fraction* $\frac{f(x)}{\psi(x)}$ *be separated into a sum of fractions of as simple types as those given in* (1)*?*

The answer is, yes.

Since the sum of several fractions has for its denominator the least common multiple of the several denominators, it follows that if $\frac{f(x)}{\psi(x)}$ can be separated into a sum of simpler fractions, the denominators of these fractions must be divisors of $\psi(x)$. Now it is known from Algebra that *every polynomial* $\psi(x)$ *having real coefficients* (and only those having real coefficients are to be considered in what follows) *can be separated into factors of either the first or the second degree, the coefficients of each factor being real.*

This fact naturally leads to the discussion of four different cases.

I. When $\psi(x)$ can be separated into real factors of the first degree, no two alike.

E.g., $\psi(x) = (x - a)(x - b)(x - c)$.

II. When the real factors are all of the first degree, some of which are repeated.

E.g., $\psi(x) = (x - a)(x - b)^2(x - c)^3$.

III. When some of the factors are necessarily of the second degree, but no two such are alike.

E.g., $\psi(x) = (x^2 + a^2)(x^2 + x + 1)(x - b)(x - c)^2$.

IV. When second degree factors occur, some of which are repeated.

E.g., $\psi(x) = (x^2 + a^2)^2(x^2 - x + 1)(x - b)$.

133. CASE I. **Factors of the first degree, none repeated.** When $\psi(x)$ is of the form

$$\psi(x) = (x-a)(x-b)(x-c)\cdots(x-n),$$

assume

$$\frac{f(x)}{\psi(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \cdots + \frac{N}{x-n},$$

in which A, B, C, $\cdots$, N are constants whose values are to be determined on condition that the sum of the terms in the right-hand member shall be identical with the left-hand member.

Ex. 1. $\int \frac{x^3 - 3x^2 + x}{x^2 - 3x + 2}\,dx.$

Dividing numerator by denominator, $\frac{x^3 - 3x^2 + x}{x^2 - 3x + 2} = x - \frac{x}{x^2 - 3x + 2}.$

Assume $\qquad \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$

By clearing of fractions,

(1) $\qquad x = A(x-2) + B(x-1).$

In order for the two members of this equation to be identical it is necessary that the coefficients of like powers of x be the same in each.

Hence $\qquad 1 = A + B, \quad 0 = -2A - B,$

from which $\qquad A = -1, \quad B = 2.$

Accordingly the given integral becomes

$$\int\left(x + \frac{1}{x-1} - \frac{2}{x-2}\right)dx = \frac{x^2}{2} + \log(x-1) - 2\log(x-2) + C$$
$$= \frac{x^2}{2} + \log\frac{x-1}{(x-2)^2} + C$$

A shorter method of calculating the coefficients can be used. Since equation (1) is an identity, it is true for all values of x. By giving x the value $x = 1$ the equation reduces to $1 = A(-1)$, or $A = -1$. Again, assume $x = 2$. Whence $2 = B$.

Ex. 2. $\int\frac{dx}{x^2-a^2}.$

Ex. 3. $\int\frac{1-3x}{x^3-x}dx.$

Ex. 4. $\int\frac{(x^3-12)\,dx}{x^2+4x+3}.$

Ex. 5. $\int\frac{(x^2-ab)\,dx}{(x-a)(x-b)}.$

Ex. 6. $\int\frac{x\,dx}{x^2-4x+1}.$

Ex. 7. $\int\frac{(x^2-1)\,dx}{(x^2-4)(4x^2-1)}.$

Ex. 8. $\int\frac{x^2-2cx+ac-ab+bc}{(x-a)(x-b)(x-c)}dx.$

Ex. 9. $\int x^2(x+a)^{-1}(x+b)^{-1}dx.$

Ex. 10. $\int\frac{(3x+1)\,dx}{2x^2+3x-2}.$

Ex. 11. $\int\frac{(x^2+ab)\,dx}{x(x-a)(x+b)}.$ OK

Ex. 12. $\int\frac{(x+4)\,dx}{2x-x^2-x^3}.$

Ex. 13. $\int\frac{x^3\,dx}{x^2+7x+12}.$ OK

Ex. 14. $\int\frac{dx}{a^2x^2-b^2}.$ OK

Ex. 15. $\int\frac{(x^2-x-1)\,dx}{(x^2-3x+2)\sqrt{2-x^2}}.$

[Separate $\frac{x^2-x-1}{x^2-3x+2}$ into partial fractions.]

Ex. 16. $\int\frac{(x^2+4x+5)\,dx}{(x^2+5x+6)\sqrt{x^2+4x+7}}.$

134. CASE II. Factors of the first degree, some repeated.

Ex. 1. $\int\frac{(5x^2-3x+1)\,dx}{x(x-1)^3}.$

Assume

$$(1)\qquad \frac{5x^2-3x+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}.$$

To justify this assumption, observe that:

(*a*) In adding the fractions in the right-hand member, the least common multiple of the denominators will be $x(x-1)^3$, which is identical with the denominator in the left-hand member.

(*b*) Further, the expressions x, $x-1$, $(x-1)^2$, $(x-1)^3$ are the only ones which can be assumed as denominators of the partial fractions, since these are the only divisors of $x(x-1)^3$.

(*c*) When equation (1) is cleared of fractions, and the coefficients of like powers of x in both members are equated, four equations are obtained, which is exactly the right number from which to determine the four unknown constants A, B, C, D.

Instead of the method just indicated in (*c*) for calculating the coefficients, a more rapid process would be as follows:

By clearing of fractions, the identity (1) may be written

$$5x^2-3x+1=A(x-1)^3+Bx(x-1)^2+Cx(x-1)+Dx.$$

Putting $x = 1$ gives at once

$$3 = D.$$

Substitute for D the value just found, and transpose the corresponding term. This gives

$$5x^2 - 6x + 1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1).$$

It can be seen by inspection that the right-hand member of the result is divisible by $x - 1$. As this relation is an identity, it follows that the left-hand member is also divisible by $x - 1$. When this factor is removed from both members, the equation reduces to

$$5x - 1 = A(x-1)^2 + Bx(x-1) + Cx.$$

Now put $x = 1$. Then

$$C = 4.$$

Substitute the value found for C, transpose, and divide by $x - 1$. The result is

$$1 = A(x-1) + Bx.$$

By giving x the values 0 and 1 in succession, it is found that

$$A = -1, \qquad B = 1.$$

Accordingly,

$$\int \frac{(5x^2 - 3x + 1)\,dx}{x(x-1)^3} = \int \left(-\frac{1}{x} + \frac{1}{x-1} + \frac{4}{(x-1)^2} + \frac{3}{(x-1)^3} \right) dx$$

$$= \log \frac{x-1}{x} - \frac{8x-5}{2(x-1)^2}.$$

Ex. 2. $\displaystyle\int \frac{dx}{(x-1)^2(x+1)}.$

Ex. 3. $\displaystyle\int \frac{(x^2 - 11x + 26)\,dx}{(x-3)^2}.$

Ex. 4. $\displaystyle\int \frac{x\,dx}{(x^2 - a^2)^2}.$

Ex. 5. $\displaystyle\int \frac{(\sqrt{2}\,x + 1)\,dx}{x^2(x + \sqrt{2})^2}.$

Ex. 6. $\displaystyle\int \frac{x^5 - 5x - 3}{x^2(x+1)^2}\,dx.$

Ex. 7. $\displaystyle\int \frac{2(x^3 + a^2x)\,dx}{x^4 - 2a^2x^2 + a^4}.$

Ex. 8. $\displaystyle\int \frac{\sqrt{2}\,dx}{(2 + \sqrt{2} - \sqrt{2}\,x)^3}.$

Ex. 9. $\displaystyle\int \frac{ax^3 + a^2x^2 + (a+1)x + a}{x^2(a+x)}\,dx.$

Ex. 10. $\displaystyle\int \frac{(x^3 - 1)\,dx}{x^3 + 3x^2}.$

Ex. 11. $\displaystyle\int (ax^2 + bx^3)^{-1}\,dx.$

Ex. 12. $\displaystyle\int \frac{(x^3 - x^2 - 1)\,dx}{(x-1)^2\sqrt{x^2 - 2x + 2}}.$

[Separate $\dfrac{x^3 - x^2 - 1}{(x-1)^2}$ into partial fractions.]

Ex. 13. $\displaystyle\int \frac{x^2}{(x-a)^3}\,dx.$

[Substitute $x - a = z$.]

135. CASE III. Occurrence of quadratic factors, none repeated.

Ex. 1. $\int \frac{(4x^2+5x+4)\,dx}{(x^2+1)(x^2+2x+2)}$.

Assume

$$(1)\qquad \frac{4x^2+5x+4}{(x^2+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2x+2}.$$

Then

$$(2)\quad 4x^2+5x+4 = (Ax+B)(x^2+2x+2) + (Cx+D)(x^2+1).$$

By equating coefficients of like powers of x

$$0 = A + C, \qquad 5 = 2A + 2B + C,$$
$$4 = 2A + B + D, \qquad 4 = 2B + D,$$

from which

$$A = 1,\ B = 2,\ C = -1,\ D = 0.$$

Hence the given integral becomes

$$\int \frac{(x+2)dx}{x^2+1} - \int \frac{x\,dx}{x^2+2x+2} = 2\tan^{-1}x + \tan^{-1}(x+1) + \tfrac{1}{2}\log\frac{x^2+1}{x^2+2x+2}.$$

To make clear the reasons for the assumption which was made concerning the form of equation (1), observe that since the factors of the denominator in the left member are x^2+1 and x^2+2x+2, these must necessarily be the denominators in the right member. Also, since the numerator of the given fraction is of lower degree than its denominator, the numerator of each partial fraction must be of lower degree than its denominator. As the latter is of the second degree in each case, the most general form for a numerator fulfilling this requirement (*i.e.*, to be of lower degree than its denominator) is an expression of the first degree such as $Ax+B$, or $Cx+D$.

Notice, besides, that in equating the coefficients of like powers of x in opposite members of equation (2), four equations are obtained which exactly suffice to determine the four unknown coefficients A, B, C, D.

Ex. 2. $\int \frac{4\,dx}{x^3+4x}$.

Ex. 3. $\int \frac{x\,dx}{(x+1)(x^2+1)}$.

Ex. 4. $\int \frac{dx}{x^8+a^8}$.

Ex. 5. $\int \frac{(a^2-b^2)\,dx}{(x^2+a^2)(x^2+b^2)}$.

Ex. 6. $\int \frac{(4x-6)\,dx}{x^4+2x^2}$.

Ex. 7. $\int \frac{x\,dx}{x^4+x^2+1}$.

Ex. 8. $\int \frac{x\,dx}{(x-a)^2(x^2+a^2)}$.

Ex. 9. $\int \frac{(x^8+2x+2)\,dx}{(x-1)(x^2+2x+2)}$.

136. CASE IV. **Occurrence of quadratic factors, some repeated.** This case bears the same relation to Case III that Case II bears to Case I, and an exactly analogous mode of procedure is to be followed.

Ex. 1. $\int \frac{2x^5 - x^4 + 8x^3 + 4}{(x^2+2)^3}dx.$

Assume

$$(1) \qquad \frac{2x^5 - x^4 + 8x^3 + 4}{(x^2+2)^3} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}.$$

Whence, by clearing of fractions,

$$2x^5 - x^4 + 8x^3 + 4 = (Ax+B)(x^2+2)^2 + (Cx+D)(x^2+2) + Ex + F.$$

Instead of equating coefficients of like powers of x, as might be done, the following method of calculating the values of A, B, C, $\cdots$ is briefer.

Substitute for x^2 the value -2, or, what is the same thing, let $x = \sqrt{-2}$. This causes all the terms of the right member to drop out except the last two, and equation (1) reduces to

$$-8\sqrt{-2} = E\sqrt{-2} + F.$$

By equating real and imaginary terms in both members,

$$-8 = E, \quad 0 = F.$$

Substitute the values found for E and F in (1), and transpose the corresponding terms. Both members will then contain the factor x^2+2. On striking this out the equation reduces to

$$2x^3 - x^2 + 4x + 2 = (Ax+B)(x^2+2) + Cx + D.$$

Proceed as before by putting $x^2 = -2$. Whence

$$4 = C\sqrt{-2} + D,$$

and therefore $\qquad 0 = C, \quad 4 = D.$

Substitute these values, transpose, and divide by x^2+2. This gives

$$2x - 1 = Ax + B,$$

whence $\qquad A = 2, \quad B = -1.$

The given integral accordingly reduces to

$$\int \frac{2x-1}{x^2+2}dx + \int \frac{4\,dx}{(x^2+2)^2} - \int \frac{8x\,dx}{(x^2+2)^3}.$$

The first term becomes

$$\int\frac{2\,x\,dx}{x^2+2}-\int\frac{dx}{x^2+2}=\log(x^2+2)-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

The second, integrated by the method of reduction (Chap. II), gives

$$\frac{x}{x^2+2}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

Finally, by applying formula I the last term integrates immediately Hence

$$\int\frac{2\,x^5-x^4+8\,x^3+4}{(x^2+2)^3}\,dx=\log(x^2+2)+\frac{x}{x^2+2}+\frac{2}{(x^2+2)^2}.$$

Ex. 2. $\int\left(\frac{x-1}{x^2+1}\right)^2dx.$

Ex. 3. $\int\frac{(x+a)^2+a^2}{(x^2+a^2)^2}\,dx.$

Ex. 4. $\int\frac{2\,x\,dx}{(1+x)(1+x^2)^2}.$

Ex. 5. $\int\frac{(-3\,x+2)\,dx}{x^2(x^2+1)^2}.$

Ex. 6. $\int\frac{2\,x^3+2\,a^2x-x^2}{(x^2+a^2)^2}\,dx.$

Ex. 7. $\int\frac{x^5\,dx}{(1+x^2)^3}.$

The principles used in the preceding cases in the assumption of the partial fractions may be summed up as follows:

Each of the denominators of the partial fractions contains one and only one prime factor of the given denominator. When a repeated prime factor occurs, all of its different powers must be used as denominators of the partial fractions.

The numerator of each of the assumed fractions is of degree one lower than the degree of the prime factor occurring in the corresponding denominator.

137. General theorem. — Since every rational fraction can be integrated by first separating, if necessary, into simpler fractions in accordance with some one of the cases considered above, the important conclusion is at once deducible :

The integral of every rational fraction can be found, and is expressible in terms of algebraic, logarithmic, and inverse-trigonometric functions.

CHAPTER IV

INTEGRATION BY RATIONALIZATION

At the end of the preceding chapter it was remarked that every rational algebraic function can be integrated. The question as to the possibility of integrating irrational functions has next to be considered. This has already been touched upon in Chapter II, where a certain type of irrational functions was treated by the method of reduction.

In the present chapter it is proposed to consider the simplest cases of irrational functions, viz., those containing $\sqrt[n]{ax+b}$ and $\sqrt{ax^2+bx+c}$, and to show how, by a process of rationalization, every such function can be integrated.

138. Integration of functions containing the irrationality $\sqrt[n]{ax+b}$. When the integrand contains $\sqrt[n]{ax+b}$, that is, the nth root of an expression of the first degree in x, but no other irrationality, it can be reduced to a rational form by means of the substitution

$$\sqrt[n]{ax+b}=z.$$

Ex. 1. $\displaystyle\int\frac{dx}{\sqrt{2x+3}-1}$.

Assume $\qquad \sqrt{2x+3}=z,$

that is, $\qquad 2x+3=z^2.$

Then $\qquad dx=z\,dz,$

and
$$\int\frac{dx}{\sqrt{2x+3}-1}=\int\frac{z\,dz}{z-1}=z+\log(z-1)$$
$$=\sqrt{2x+3}+\log(\sqrt{2x+3}-1).$$

Ex. 2. $\int \frac{1 + x^{\frac{1}{6}} - x^{\frac{1}{3}} - \sqrt{x}}{x^{\frac{2}{3}} + x}\,dx.$

It would appear at first sight that this integrand contains several irrationalities, viz., $\sqrt{x}$, $\sqrt[3]{x}$, $\sqrt[6]{x}$. It is readily seen, however, that they are all powers of $\sqrt[6]{x}$, and hence the substitution $\sqrt[6]{x} = z$ will rationalize the expression to be integrated.

ok **Ex. 3.** $\int \frac{dx}{x\sqrt{x+1}}.$ $\sqrt{x+1} = z$

Ex. 4. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$

Ex. 5. $\int \frac{dx}{2\sqrt{x-1} + x}.$

ok **Ex. 6.** $\int \frac{dx}{(x-1)\sqrt{x-2}}.$ let $\sqrt{x-2} = z$

Ex. 7. $\int \frac{dx}{(x-a-b^2)\sqrt{x-a}}.$ let $\sqrt{x-a}$

Ex. 8. $\int \frac{x^{\frac{1}{7}} + \sqrt{x}}{x^{\frac{8}{7}} + x^{\frac{15}{14}}}\,dx.$

When two irrationalities of the form $\sqrt{ax+b}$, $\sqrt{cx+d}$ occur in the integrand, the first radical can be made to disappear by the substitution

$$\sqrt{ax+b} = z.$$

The second radical then reduces to

$$\sqrt{\frac{c}{a}(z^2 - b) + d},$$

and the method of the next article can be applied.

here

139. Integration of expressions containing $\sqrt{ax^2 + bx + c}$. Every expression containing $\sqrt{ax^2 + bx + c}$, but no other irrationality, can be rationalized by a proper substitution. In order to make the necessary steps clearer, a geometrical interpretation of the problem will be very useful.

To this end let the given radical be represented by y; that is, let

$$y^2 = ax^2 + bx + c. \tag{1}$$

If now (x, y) be regarded as the rectangular coördinates of a point in a plane, equation (1) represents a conic (Fig. 60).

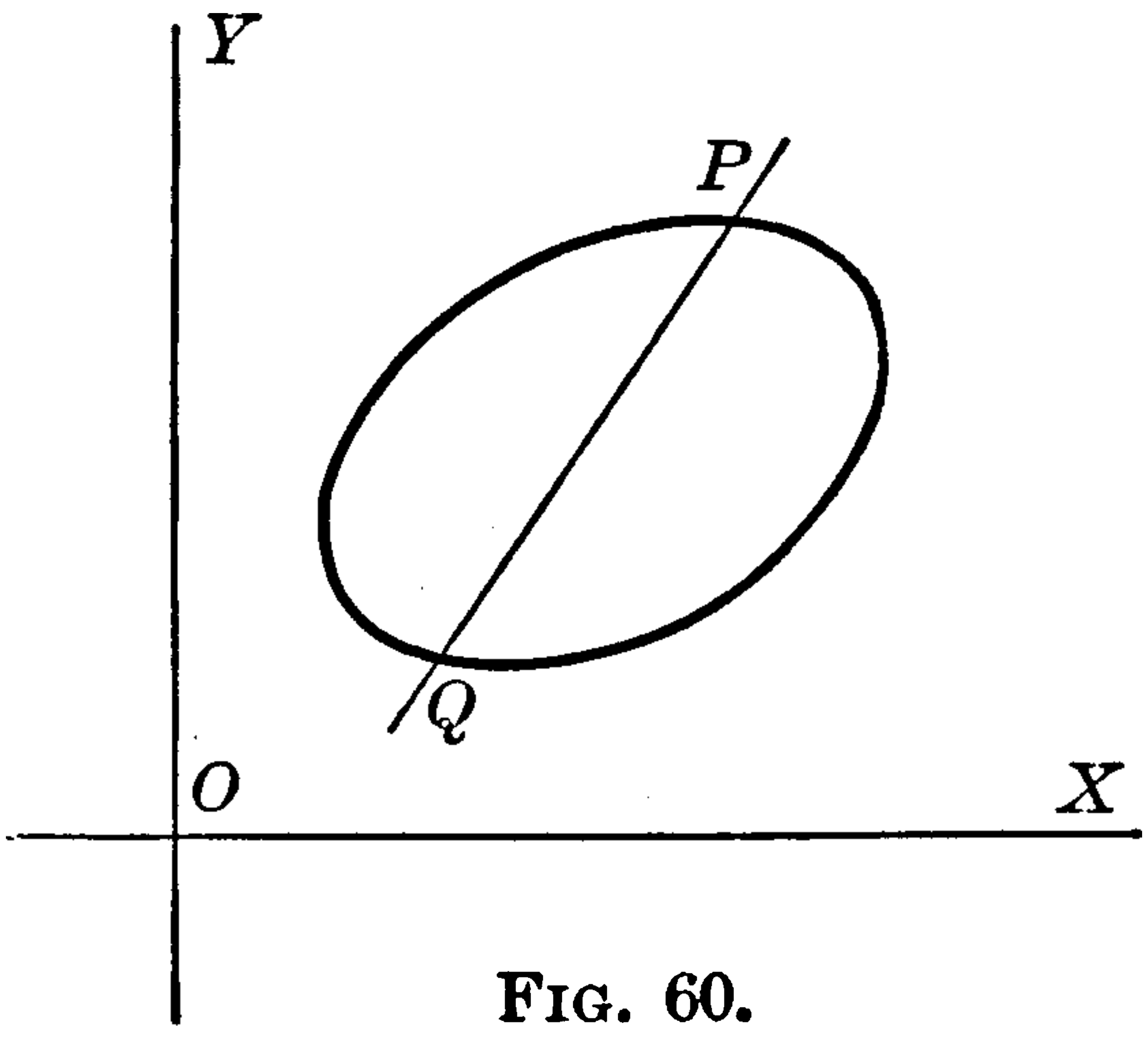

FIG. 60.

Let (h, k), or Q, be a given point on this curve. The equation of any line through this point is

$$y-k=z(x-h), \qquad (2)$$

in which z is the slope of the line. The line (2) will intersect the conic in a second point P. It is geometrically evident that the coördinates (x, y) of P depend on the value of z, and in such a way that to each value of z corresponds only one pair of values x, y.

Consequently the variables x and y can be rationally expressed in terms of the variable z. This is done by treating equations (1) and (2) as simultaneous, and solving for x and y in terms of z.

For example, suppose it were desired to rationalize an expression containing $\sqrt{x^2-5x+8}$.

Let $$y^2=x^2-5x+8,$$

and select (1, 2) for the point Q.

Then $$y-2=z(x-1)$$

represents any line passing through Q. In solving these two equations simultaneously for x and y, the elimination of y gives

$$z^2(x-1)^2+4z(x-1)=x^2-5x+4.$$

This quadratic equation in x has two roots, one of which should be $x=1$, since this is the value of x at Q, one of the points of intersection. The other root, corresponding to the variable point P, is

$$x = \frac{z^2 - 4z - 4}{z^2 - 1}.$$

From this follows

$$y - 2 = z(x - 1) = z\left(\frac{z^2 - 4z - 4}{z^2 - 1} - 1\right),$$

or
$$y = \sqrt{x^2 - 5x + 8} = \frac{-2z^2 - 3z - 2}{z^2 - 1}.$$

Two particular cases of the method given above deserve to be noticed.

(*a*) *When the conic intersects the x-axis.*

In this case the quadratic expression $ax^2 + bx + c$ has real factors, say,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

The conic (1) intersects the x-axis in the two points $(\alpha, 0)$ and $(\beta, 0)$, either one of which may be conveniently selected for the point Q.

The equation of any line QP through the first point is

$$y = z(x - \alpha), \tag{A}$$

and the equation of any line through the second point,

$$y = z(x - \beta). \tag{A'}$$

Either one of these equations, combined with (1), will effect the desired rationalization.

(*b*) *When the conic is an hyperbola.*

This case occurs when the coefficient of x^2 is positive. The curve extends to infinity in two different directions, namely, the directions of the asymptotes. If one of the points at infinity on the curve be taken for the point Q, the lines QP passing through this point are parallel to that asymptote which touches the curve at Q. The equations of the asymptotes are

$$y = \pm\sqrt{a}\left(x + \frac{b}{2a}\right).$$

Accordingly the lines parallel to the one asymptote are

$$y = \sqrt{a}\,x + z, \tag{B}$$

and those parallel to the other

$$y = -\sqrt{a}\,x + z. \tag{B'}$$

Either of these equations used in place of (2) will serve equally well in expressing x and $y(=\sqrt{ax^2+bx+c})$ rationally in terms of a new variable z.

Ex. 1. $\displaystyle\int\frac{dx}{(x+\sqrt{x^2+2x-1})^2}.$

The conic $y=\sqrt{x^2+2x-1}$ is an hyperbola and formula (B) can be applied. This gives

$$\sqrt{x^2+2x-1} = x + z,$$

whence by squaring and solving for x,

$$x = \frac{z^2+1}{2(1-z)},$$

and accordingly

$$dx = \frac{-z^2+2z+1}{2(1-z)^2}\,dz,$$

$$\sqrt{x^2+2x-1} = \frac{-z^2+2z+1}{2(1-z)}.$$

When these expressions are substituted in the given integral, it becomes

$$\int\frac{-z^2+2z+1}{2(1+z)^2}\,dz = \tfrac{1}{2}\int\left(-1+\frac{4}{1+z}-\frac{2}{(1+z)^2}\right)dz$$

$$= \tfrac{1}{2}\left[-z+4\log(1+z)+\frac{2}{1+z}\right]$$

$$= \tfrac{1}{2}(x-\sqrt{x^2+2x-1}) + \frac{1}{1-x+\sqrt{x^2+2x-1}} + 2\log\left[1+\sqrt{x^2+2x-1}-x\right].$$

Since the conic $y=\sqrt{x^2+2x-1}$ cuts the x-axis, formula (A) [or (A')] could be used for the purpose of rationalization.

Ex. 2. $\int \frac{\sqrt{1+x}\,dx}{(1-x)\sqrt{1-x}}.$

The denominator being rationalized, the integrand takes the form

$$\frac{\sqrt{1-x^2}}{(1-x)^2}.$$

The conic

$$y = \sqrt{1-x^2}$$

intersects the x-axis in two points $(\pm 1, 0)$.

If the point $(1, 0)$ be chosen for Q, the equation of any line passing through this point is

$$y = z(x-1).$$

The simultaneous solution of these two equations gives

$$x = \frac{z^2-1}{z^2+1}, \quad y = \frac{-2z}{z^2+1},$$

whence

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{(1-x)^2}\,dx &= \int \frac{-2z^2\,dz}{z^2+1} \\ &= 2(-z + \tan^{-1} z) \\ &= 2\left(\frac{-\sqrt{1-x^2}}{x-1} + \tan^{-1}\frac{\sqrt{1-x^2}}{x-1}\right).\end{aligned}$$

Ex. 3. $\int \frac{dx}{(1-x)(1-\sqrt{1-x^2})}.$

Ex. 4. $\int \frac{dx}{\sqrt{2x^2-3x+1}\,[\sqrt{2x^2-3x+1}+\sqrt{2}(x-1)]}.$

140. From what precedes, combined with the theorem of Art. 137, it follows that every rational function depending only on x and the square root of a polynomial of not higher than the second degree in x can be integrated, and the result expressed in terms of known functions.

EXERCISES ON CHAPTER IV

1. $\int \frac{(-x^3+4x)\,dx}{(x^2+2)^2\sqrt{x^2-1}}.$

[Substitute $\sqrt{x^2-1} = z$.]

2. $\int \frac{[(x-a)^{\frac{2}{3}}-1]\,dx}{2(x-a)^{\frac{4}{3}}-(x-a)^{\frac{2}{3}}}.$

3. $\int \frac{(x^4+x^2-1)\,dx}{(x^2+1)^2(x^2+2)^{\frac{3}{2}}}.$

4. $\displaystyle\int\frac{dx}{x+\sqrt{x-1}}$.

5. $\displaystyle\int\frac{dx}{x+\sqrt{x^2-1}}$.

6. $\displaystyle\int\frac{dx}{\sqrt{x^2+2}+\sqrt{x^2+1}}$.

7. $\displaystyle\int\frac{x\,dx}{(a+x)^{\frac{1}{3}}}$.

8. $\displaystyle\int\frac{(2-3\,x^{-\frac{1}{6}})\,dx}{x-3\,x^{\frac{5}{6}}+5\,x^{\frac{2}{3}}}$.

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9. $\displaystyle\int\frac{dx}{\sqrt{x^2-1}\sqrt{\sqrt{x+1}+\sqrt{x-1}}}$.

[Assume $\sqrt{x+1}+\sqrt{x-1}=z$.]

10. $\displaystyle\int\frac{dx}{(x^2+a^2)\sqrt{x^2-a^2}}$.

[Assume $x=a\sec\theta$.]

11. $\displaystyle\int\frac{1+\sqrt{x}}{1+\sqrt[3]{x}}\,dx$.

12. $\displaystyle\int\sqrt[3]{\frac{1-x}{1+x}}\,\frac{dx}{(1+x)^2}$.

$\left[\text{Substitute } \dfrac{1-x}{1+x}=z^3.\right]$

CHAPTER V

INTEGRATION OF TRIGONOMETRIC AND OTHER TRANSCENDENTAL FUNCTIONS

141. In regard to the integration of trigonometric functions, it is to be remarked in the first place that every rational trigonometric function can be rationally expressed in terms of sine and cosine.

It is accordingly evident that such functions can be integrated by means of the substitution

$$\sin x = z.$$

After the substitution has been effected, the integrand may involve the irrationality

$$\sqrt{1 - z^2} \, (= \cos x).$$

This can be removed by rationalization, as explained in the preceding chapter, or the method of reduction may be employed.

The substitution $\cos x = z$ will serve equally well.

It is usually easier, however, to integrate the trigonometric forms without any such previous transformation to algebraic functions. The following articles treat of the cases of most frequent occurrence.

142. $\int \sec^{2n} x \, dx, \quad \int \operatorname{cosec}^{2n} x \, dx.$

In this case n is supposed to be a positive integer.

If $\sec^{2n} x \, dx$ be written in the form

$$\sec^{2n-2} x \cdot \sec^2 x \, dx = (1 + \tan^2 x)^{n-1} d(\tan x),$$

the first integral becomes

$$\int (\tan^2 x + 1)^{n-1} d(\tan x).$$

If $(\tan^2 x + 1)^{n-1}$ be expanded by the binomial formula and integrated term by term, the required result is readily obtained.

In like manner,

$$\int \operatorname{cosec}^{2n} x\, dx = \int \operatorname{cosec}^{2n-2} x \cdot \operatorname{cosec}^2 x\, dx$$
$$= -\int (\cot^2 x + 1)^{n-1} d(\cot x).$$

This last form can be integrated, as in the preceding case, by expanding the binomial in the integrand.

The same method will evidently apply to integrals of the form

$$\int \tan^m x \sec^{2n} x\, dx, \quad \int \cot^m x \operatorname{cosec}^{2n} x\, dx,$$

in which m is any number.

EXERCISES

1. $\int \frac{dx}{\cos^4 x}$.
2. $\int \operatorname{cosec}^4 x\, dx$.
3. $\int \sec^6 x\, dx$.
4. $\int \frac{dx}{\sin^8 x \cos^8 x}$.
5. $\int \frac{(1 - \cos x)^2 dx}{\sin^4 x}$.
6. $\int \frac{dx}{\sin^4 x \cos^4 x (\cos^4 x - \sin^4 x)^4}$.
7. $\int \frac{dx}{\sin^3 x \cos x} \left(= \int \tan^{-3} x \sec^4 x\, dx\right)$.
8. $\int \frac{\cos^2 x\, dx}{\sin^6 x}$.

143. $\int \sec^m x \tan^{2n+1} x\, dx$, $\int \operatorname{cosec}^m x \cot^{2n+1} x\, dx$.

In these integrands n is a positive integer, or zero, so that $2n + 1$ is any positive odd integer, while m is unrestricted.

The first integral may be written in the form

$$\int \sec^{m-1}x \tan^{2n}x \cdot \sec x \tan x\,dx$$
$$= \int \sec^{m-1}x\,(\sec^2 x - 1)^n d(\sec x),$$

which can be integrated after expanding $(\sec^2 x - 1)^n$ by the binomial formula.

Similarly,

$$\int \operatorname{cosec}^m x \cot^{2n+1}x\,dx = \int \operatorname{cosec}^{m-1}x \cot^{2n}x \cdot \operatorname{cosec} x \cot x\,dx$$
$$= -\int \operatorname{cosec}^{m-1}x\,(\operatorname{cosec}^2 x - 1)^n d(\operatorname{cosec} x).$$

EXERCISES

1. $\int \sec^2 x \tan^3 x\,dx.$
2. $\int \operatorname{cosec}^3 x \cot^5 x\,dx.$
3. $\int \frac{\sec ax}{\cot^5 ax}\,dx.$
4. $\int \sin x \cot^3 x\,dx.$
5. $\int \tan^5 x\,dx.$
6. $\int \frac{\sin^3 x\,dx}{\cos^n x} \left(= \int \sec^{n-3}x \tan^3 x\,dx\right).$
7. $\int \tan x\,dx.$
8. $\int \cot x\,dx.$

144. $\int \tan^n x\,dx$, $\int \cot^n x\,dx$.

The first integral can be treated thus:

$$\int \tan^n x\,dx = \int \tan^{n-2} \cdot \tan^2 x\,dx$$
$$= \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx$$
$$= \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx.$$

When n is a positive integer, the exponent of $\tan x$ may be diminished by successive applications of this formula until it becomes zero (when n is even), or one (when n is odd).

In like manner,

$$\int \cot^n x\,dx = \int \cot^{n-2} x \cot^2 x\,dx$$
$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\,dx$$
$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx.$$

Since $\tan x$ and $\cot x$ are reciprocals of each other, the above method is sufficient to integrate any integer power of $\tan x$, or $\cot x$.

Another method of procedure would be to make the substitution $\tan x = z$, whence

$$\int \tan^n x\,dx = \int \frac{z^n\,dz}{1+z^2}.$$

If the exponent n is a fraction, say $n = \frac{p}{q}$, the last integral can be rationalized by the substitution $z = u^q$.

It is evident from this that any rational power of tangent or cotangent can be integrated.

EXERCISES

1. $\int \cot^4 x\,dx.$

2. $\int \tan^3 ax\,dx.$

3. $\int (\tan x - \cot x)^3\,dx.$

4. $\int (\tan^n x + \tan^{n-2} x)\,dx.$

5. $\int \tan^8 x\,dx.$

When n is a positive integer show that

6. $\int \tan^{2n} x\,dx = \frac{\tan^{2n-1} x}{2n-1} - \frac{\tan^{2n-3} x}{2n-3} + \cdots + (-1)^{n-1}(\tan x - x).$

7. $\int \tan^{2n+1} x\,dx = \frac{\tan^{2n} x}{2n} - \frac{\tan^{2n-2} x}{2n-2} + \cdots + (-1)^{n-1}(\frac{1}{2}\tan^2 x + \log \cos x).$

145. $\int \sin^m x \cos^n x \, dx$.

(*a*) Either *m* or *n* a positive odd integer.

If one of the exponents, for example m, is a positive odd integer, the given integral may be written

$$\int \sin^{m-1} x \cos^n x \sin x \, dx = -\int (1 - \cos^2 x)^{\frac{m-1}{2}} \cos^n x \, d(\cos x).$$

Since m is odd, $m - 1$ is even, and therefore $\frac{m-1}{2}$ is a positive integer. Hence the binomial can be expanded into a finite number of terms, and thus the integration can be easily completed.

Ex. **1.** $\int \sin^5 x \sqrt{\cos x} \, dx$.

According to the method just indicated this integral can be reduced to

$$-\int \sin^4 x \sqrt{\cos x} \, d(\cos x) = -\int (1 - \cos^2 x)^2 (\cos x)^{\frac{1}{2}} d(\cos x)$$

$$= -\tfrac{2}{3} \cos^{\frac{3}{2}} x + \tfrac{4}{7} \cos^{\frac{7}{2}} x - \tfrac{2}{11} \cos^{\frac{11}{2}} x.$$

Ex. **2.** $\int \sin^3 x \, dx$.

Ex. **3.** $\int \sin^3 x \cos^4 x \, dx$.

Ex. **4.** $\int \frac{\cos^5 x}{\sin x} dx$.

Ex. **5.** $\int \frac{\sin^5 x \, dx}{\cos^2 x \sqrt[3]{\cos x}}$.

Ex. **6.** $\int \frac{\sin^3 x \, dx}{\sqrt{1 - \cos x}}$.

(*b*) $m + n$ an even negative integer.

In this case the integral may be put in the form

$$\int \frac{\sin^m x}{\cos^m x} \cos^{m+n} x \, dx = \int \tan^m x \sec^{-(m+n)} x \, dx,$$

which can be integrated by Art. 142, since the exponent $-(m + n)$ of $\sec x$ is an even positive integer.

Ex. **7**. $\int \frac{\sqrt{\sin x}}{\cos^{\frac{9}{2}} x}\, dx.$

The integration is effected in the following steps:

$$\begin{aligned}\int \frac{\sqrt{\sin x}\, dx}{\sqrt{\cos x}\, \cos^4 x} &= \int \tan^{\frac{1}{2}} x \sec^4 x\, dx \\ &= \int \tan^{\frac{1}{2}} x\, (\tan^2 x + 1)\, d\,(\tan x) \\ &= 2 \tan^{\frac{3}{2}} x\, (\tfrac{1}{3} + \tfrac{1}{7} \tan^2 x).\end{aligned}$$

Ex. **8**. $\int \frac{\cos^2 x}{\sin^4 x}\, dx.$

Ex. **9**. $\int \frac{dx}{\sin^6 x}.$

Ex. **10**. $\int \frac{\cos^4 x}{\sin^8 x}\, dx.$

Ex. **11**. $\int \frac{dx}{\sin^4 x \cos^2 x}.$

Ex. **12**. $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}.$

Ex. **13**. $\int \frac{\sin^{n-2} x}{\cos^{n+2} x}\, dx.$

(*c*) **Multiple angles.**

When m and n are both even positive integers, integration may be effected by the use of multiple angles. The trigonometric formulas used for this purpose are

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin x \cos x = \frac{\sin 2x}{2}.$$

Ex. **14**. $\int \sin^2 x \cos^4 x\, dx.$

$$\begin{aligned}\int \sin^2 x \cos^4 x\, dx &= \int (\sin x \cos x)^2 \cos^2 x\, dx \\ &= \int \frac{\sin^2 2x}{4} \frac{1 + \cos 2x}{2}\, dx \\ &= \tfrac{1}{8} \int \sin^2 2x\, dx + \tfrac{1}{16} \int \sin^2 2x \cos 2x\, d\,(2x) \\ &= \tfrac{1}{8} \int \frac{1 - \cos 4x}{2}\, dx + \tfrac{1}{16} \frac{\sin^3 2x}{3} \\ &= \tfrac{1}{16} x - \tfrac{1}{64} \sin 4x + \tfrac{1}{48} \sin^3 2x.\end{aligned}$$

Ex. 15. $\int \cos^2 x \sin^2 x \, dx.$

Ex. 16. $\int \sin^2 x \cos^6 x \, dx.$

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Ex. 17. $\int \sin^4 x \cos^4 x \, dx.$

Ex. 18. $\int (\sin^4 x - \cos^4 x)^4 \, dx.$

Integrate the two following by the aid of multiple angles.

Ex. 19. $\int \frac{dx}{\sin^4 x \cos^4 x}.$

Ex. 20. $\int \frac{dx}{\sin x \cos^3 x - \sin^3 x \cos x}.$

Integrate the following by any of the preceding methods.

Ex. 21. $\int \frac{\sin^4 x}{\cos^2 x} dx = \int \frac{(1 - \cos^2 x)^2}{\cos^2 x} dx = \int (\sec^2 x - 2 + \cos^2 x) \, dx.$

Ex. 22. $\int \frac{\cos^6 x \, dx}{\sin^4 x}.$

✓ **Ex. 23.** $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx.$

[Substitute $x = a \sin \theta$.]

Ex. 24. $\int x^2 \sqrt{a^2 - x^2} \, dx.$

Ex. 25. $\int \frac{dx}{x^3 \sqrt{x^2 - a^2}}.$

[Substitute $x = a \sec \theta$.]

146. $\int \frac{dx}{a + b \cos x}, \int \frac{dx}{a + b \sin x}.$

Write $a + b \cos x$ in the form

$$a\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) + b\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)$$

$$= (a + b) \cos^2 \frac{x}{2} + (a - b) \sin^2 \frac{x}{2}$$

$$= (a - b) \cos^2 \frac{x}{2}\left(\frac{a + b}{a - b} + \tan^2 \frac{x}{2}\right).$$

Then $$\int \frac{dx}{a + b \cos x} = \frac{2}{a - b} \int \frac{\sec^2 \frac{x}{2} \, d\left(\frac{x}{2}\right)}{\frac{a + b}{a - b} + \tan^2 \frac{x}{2}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{\tan \frac{x}{2}}{\sqrt{\frac{a + b}{a - b}}}\right].$$

This result has a real form provided that a is numerically greater than b, since $\frac{a+b}{a-b}$ and $a^2 - b^2$ are then positive.

When a is numerically less than b the integral may be written

$$\int \frac{dx}{a + b\cos x} = \frac{2}{a-b}\int \frac{\sec^2\frac{x}{2}\,d\left(\frac{x}{2}\right)}{\tan^2\frac{x}{2} - \frac{b+a}{b-a}}$$

$$= \frac{-1}{\sqrt{b^2-a^2}} \log \frac{\tan\frac{x}{2} - \sqrt{\frac{b+a}{b-a}}}{\tan\frac{x}{2} + \sqrt{\frac{b+a}{b-a}}}.$$

The integration of

$$\int \frac{dx}{a + b\sin x}$$

is effected by making the substitution

$$x = y + \frac{\pi}{2},$$

which reduces the integral to

$$\int \frac{dy}{a + b\cos y}.$$

The preceding results may then be applied.

EXERCISES

1. $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$.
2. $\int \frac{dx}{a\sin x + b\cos x}$.

SUGGESTION. Write the denominator of Ex. 2 in the form

$$2a\sin\frac{x}{2}\cos\frac{x}{2} + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right),$$

divide numerator and denominator by $\cos^2\frac{x}{2}$, and replace $\tan\frac{x}{2}$ by a new variable.

3. $\int \frac{dx}{5+3\cos 2x}$.

4. $\int \frac{dx}{5-3\sin x}$.

5. $\int \frac{dx}{1-2\sin 2x}$.

6. $\int \frac{dx}{(a\sin x+b\cos x)^2}$.

7. $\int \frac{dx}{1+\cos^2 x}$.

147. $\int e^{ax}\sin nx\,dx$, $\int e^{ax}\cos nx\,dx$.

Integrate $\int e^{ax}\sin nx\,dx$ by parts, assuming

$$u=\sin nx, \text{ and } dv=e^{ax}\,dx.$$

This gives

$$\int e^{ax}\sin nx\,dx=\frac{1}{a}e^{ax}\sin nx-\frac{n}{a}\int e^{ax}\cos nx\,dx. \qquad (1)$$

Integrate the same expression again, assuming this time

$$u=e^{ax}, \quad dv=\sin nx\,dx.$$

Then

$$\int e^{ax}\sin nx\,dx=-\frac{1}{n}e^{ax}\cos nx+\frac{a}{n}\int e^{ax}\cos nx\,dx. \qquad (2)$$

Multiply (1) by $\frac{a}{n}$ and (2) by $\frac{n}{a}$ and add. The integrals in the right members are eliminated, and the result is

$$\int e^{ax}\sin nx\,dx=\frac{e^{ax}(a\sin nx-n\cos nx)}{a^2+n^2}.$$

By subtracting (1) from (2), the formula

$$\int e^{ax}\cos nx\,dx=\frac{e^{ax}(n\sin nx+a\cos nx)}{a^2+n^2}$$

is obtained.

EXERCISES ON CHAPTER V

1. Show that

$$(1+n)\int \sec^{n+2}x\,dx=\tan x\sec^n x+n\int \sec^n x\,dx.$$

Integrate by parts, taking $u=\sec^n x$, $dv=\sec^2 x\,dx$.

2. Show that

$$(1+n)\int \operatorname{cosec}^{n+2}x\,dx = -\cot x\operatorname{cosec}^{n}x + n\int \operatorname{cosec}^{n}x\,dx.$$

3. $\displaystyle\int\frac{\sqrt{\tan x}\,dx}{\sin x\cos x}.$

4. $\displaystyle\int\frac{dx}{(1-x)\sqrt{1-x^2}}.$

[Put $x=\cos\theta$.]

5. $\displaystyle\int\frac{\sin x\,dx}{\cos^5 x}.$

6. $\displaystyle\int\frac{dx}{\cos x\sin^2 x}.$

7. $\displaystyle\int\frac{\sin^4 x}{\cos^3 x}\,dx.$

8. $\displaystyle\int e^{\frac{x}{2}}\cos\frac{x}{2}\,dx.$

9. $\displaystyle\int\frac{\sin 2x}{e^x}\,dx.$

10. $\displaystyle\int e^{2x}\sin^2 x\,dx.$

11. $\displaystyle\int e^{x}\sin 2x\sin x\,dx.$

[SUGGESTION. $2\sin 2x\sin x=\cos x-\cos 3x$.]

12. Show that

$$\int\sin ax\sin bx\,dx=\frac{\sin(a-b)x}{2(a-b)}-\frac{\sin(a+b)x}{2(a+b)}$$

Use the trigonometric formula

$$\sin\alpha\sin\beta=\tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)].$$

13. Show that

$$\int\sin ax\cos bx\,dx=-\frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}.$$

14. Show that

$$\int\cos ax\cos bx\,dx=\frac{\sin(a-b)x}{2(a-b)}+\frac{\sin(a+b)x}{2(a+b)}.$$

15. $\displaystyle\int\sin^n x\cos^3 x\,dx.$

16. $\displaystyle\int\frac{dx}{\sin^{\frac{3}{2}}x\cos^{\frac{5}{2}}x}.$

17. $\displaystyle\int(\tan x+\cot x)^6\,dx.$

18. $\displaystyle\int\frac{dx}{(1+\cos x)^3}.$

CHAPTER VI

INTEGRATION AS A SUMMATION

148. In the preceding five chapters various methods of integration have been explained. The final object in every case has been to determine a function $F(x)$ such that its derivative should be identical with a given function $f(x)$. It is now proposed to analyze this idea a little more fully, and to show that it readily leads to a view of integration which is of the highest interest and importance.

In order to obtain the derivative of a function $F(x)$ it is necessary in the first place to determine the increment

$$F(x+\Delta x)-F(x) \tag{1}$$

which the function $F(x)$ takes when the independent variable x takes the increment Δx.

The expression (1) can be put in a form more convenient for present purposes, if it be assumed that for all values of x under consideration $F(x+\Delta x)$ can be expanded by means of Taylor's theorem [Art. 41, p. 66]. This expansion is

$$\begin{aligned}&F(x+\Delta x)\\&\quad=F(x)+F'(x)\Delta x+\frac{F''(x)}{2}(\Delta x)^2+\frac{F'''(x)}{3!}(\Delta x)^3+\cdots.\end{aligned}$$

From this, by transposing $F(x)$, the increment (1) is obtained in the form of a series, viz.,

$$\begin{aligned}&F(x+\Delta x)-F(x)\\&=F'(x)\Delta x+\Delta x\left[\frac{F''(x)}{2}\Delta x+\frac{F'''(x)}{3!}(\Delta x)^2+\cdots\right]\\&=f(x)\Delta x+\phi(x)\Delta x.\end{aligned} \tag{2}$$

In the last expression $\phi(x)$ has been written for brevity in place of the series in brackets, and $f(x)$ is the equivalent of $F'(x)$, since by supposition $f(x)$ is the derivative of $F(x)$.

Suppose now that the variable x starts with a given value a and increases until it reaches another given value b. The function $F(x)$ will change accordingly, beginning with the value $F(a)$ and ending with $F(b)$. The difference between these two, viz.,

$$F(b) - F(a)$$

can be determined by the aid of (2) in the following manner.

Let the variation of x from a to b be imagined to occur in successive steps, first from a to $a + \Delta x$, then from $a + \Delta x$ to $a + 2\,\Delta x$, and so on. The increment which the function $F(x)$ takes at the first step of the change is $F(a + \Delta x) - F(a)$. Its value is found by giving x the value a in formula (2). That is,

$$F(a + \Delta x) - F(a) = f(a)\Delta x + \phi(a)\Delta x.$$

The increment that $F(x)$ takes at the second step is

$$F(a + 2\,\Delta x) - F(a + \Delta x) = f(a + \Delta x)\Delta x + \phi(a + \Delta x)\Delta x,$$

the right member of which is found by substituting $x = a + \Delta x$ in (2). In like manner, by giving x the values $a + 2\,\Delta x$, $a + 3\,\Delta x$, ..., $a + (n-1)\Delta x$, the additional equations are found:

$$F(a + 3\,\Delta x) - F(a + 2\,\Delta x) = f(a + 2\,\Delta x)\Delta x + \phi(a + 2\,\Delta x)\Delta x,$$

$$F(a + 4\,\Delta x) - F(a + 3\,\Delta x) = f(a + 3\,\Delta x)\Delta x + \phi(a + 3\,\Delta x)\Delta x,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$F(a + n\,\Delta x) - F(a + \overline{n-1}\,\Delta x) = f(a + \overline{n-1}\,\Delta x)\Delta x \\ + \phi(a + \overline{n-1}\,\Delta x)\Delta x.$$

Assume $$a + n\,\Delta x = b, \tag{3}$$

and substitute in the first term of the preceding equation. The addition of the above n equations then gives

$$F(b) - F(a)$$
$$= \Delta x\,[f(a) + f(a+\Delta x) + f(a+2\,\Delta x) + \cdots + f(a+\overline{n-1}\,\Delta x)]$$
$$+ \Delta x\,[\phi(a) + \phi(a+\Delta x) + \phi(a+2\Delta x) + \cdots + \phi(a+\overline{n-1}\,\Delta x)].$$

This expression for $F(b) - F(a)$, while depending on the given function $f(x)$, contains also a series of successive values of $\phi(x)$, viz.,

$$\Delta x[\phi(a) + \phi(a+\Delta x) + \cdots + \phi(a+\overline{n-1}\,\Delta x)].$$

This latter can be gotten rid of by taking its limit as Δx approaches zero.

For, since

$$\phi(x) = \frac{F''(x)}{2}\Delta x + \frac{F'''(x)}{3\,!}(\Delta x)^2 + \cdots,$$

it follows that

$$\lim_{\Delta x \doteq 0} \phi(x) = 0, \tag{4}$$

and hence, if Φ denote the numerically greatest term of the series

$$\phi(a) + \phi(a+\Delta x) + \cdots,$$

then

$$\Delta x\,[\phi(a) + \phi(a+\Delta x) + \cdots]\,| \leqq |\,\Delta x[\Phi + \Phi \cdots (n \text{ terms})]$$
$$| \leqq |\,\Delta x \cdot n\Phi$$
$$| \leqq |\,(b-a)\Phi.$$

But since, on account of (4),

$$\lim_{\Delta x \doteq 0} \Phi = 0,$$

it follows that

$$\lim_{\Delta x \doteq 0} [\phi(a) + \phi(a+\Delta x) + \cdots]\,\Delta x = 0,$$

and hence

$$F(b) - F(a)$$
$$= \lim_{\Delta x \doteq 0} [f(a) + f(a+\Delta x) + \cdots + f(a+\overline{n-1}\,\Delta x)]\Delta x. \tag{5}$$

The second member of (5) is denoted for brevity by the symbol

$$\int_a^b f(x)\,dx,$$

and is called the *definite integral* of $f(x)$ between the limits a and b.

Suppose one of the limits, say the upper limit b, is regarded as variable, while the other has a fixed value. To emphasize this assumption concerning the variability of b, let it be replaced by the letter x. Then equation (5) may be written

$$F(x) = \lim_{\Delta x \doteq 0}\,[f(a) + f(a + \Delta x) + \cdots + f(a + \overline{n-1}\,\Delta x)]\,\Delta x + F(a). \qquad (6)$$

Here the term $F(a)$ has a fixed, although arbitrary, value depending on the particular choice that is made for the constant a. It may be regarded as a constant of integration.

Formula (6) expresses in two steps the solution of the problem of determining the function $F(x)$:

(1) *Find the sum of the series of n terms*

$$f(a),\ f(a + \Delta x),\ f(a + 2\,\Delta x),\ \cdots,\ f(a + (n-1)\Delta x),$$

these being the values of the given function $f(x)$ corresponding to the n equidistant values of x,

$$a,\ a + \Delta x,\ a + 2\,\Delta x,\ \cdots,\ a + (n-1)\,\Delta x.$$

(2) *Find the limit of the product of this sum by Δx, as Δx approaches zero while n increases to infinity, subject to the condition $n\Delta x = x - a$.*

The addition of an arbitrary constant of integration makes the solution the most general possible.

The method just formulated for determining the integral $F(x)$ of a given function $f(x)$ is not suitable for the actual work of integration, since, with few exceptions (cf. Exs. 1, 2 below), the summation of the series in the right-hand member of (6) presents insuperable difficulties.

On the other hand, formula (5) admits of a very simple geometrical or physical interpretation in most of the applications of the calculus, and herein lies one of its chief merits. It places before one a very convenient and useful formulation of many of the problems of geometry, mechanics, physics, etc., the final solution of which is most readily effected by the evaluation of the definite integral

$$\int_a^b f(x)\,dx$$

in the following manner. First obtain the function $F(x)$ by integrating $f(x)dx$ according to the methods already explained in the preceding chapters.

Determine $F(b)$ and $F(a)$ by substituting the limits b and a in the result. Finally subtract $F(a)$ from $F(b)$. This gives

$$\int_a^b f(x)dx = F(b) - F(a)$$

as the value of the definite integral.

Ex. 1. Given $f(x) = e^x$, find $F(x)$ by the method of summation.
For the sake of brevity write $\Delta x = h$. Then formula (6) gives

$$F(x) = \lim_{h \doteq 0}\,[e^a + e^{a+h} + e^{a+2h} + \cdots + e^{a+(n-1)h}]h + F(a).$$

The sum in the right member may be written

$$e^a[1 + e^h + e^{2h} + \cdots + e^{(n-1)h}]h = e^a\frac{1-e^{nh}}{1-e^h}\cdot h$$

(by the formula for summing a geometric series)

$$= e^a h\frac{1-e^{x-a}}{1-e^h} \qquad (nh = x - a)$$

$$= e^a(e^{x-a} - 1)\cdot\frac{h}{e^h - 1}.$$

As h is made to approach zero the factor $\frac{h}{e^h - 1}$ becomes indeterminate. Its limit is found by the method of Chapter V (p. 77) to be

$$\lim_{h \doteq 0} \frac{h}{e^h - 1} = 1.$$

Hence $$\lim_{h \doteq 0} e^a [1 + e^h + \cdots + e^{(n-1)h}]h = e^x - e^a,$$

and accordingly $F(x) = \int e^x dx = e^x - e^a + F(a) = e^x + C,$

in which $C (= F(a) - e^a)$ may be regarded as an arbitrary constant of integration.

Ex. 2. Given $f(x) = ax$, find $\int ax\,dx$ by the method of summation.

149. Geometrical interpretation of the definite integral as an area. Let the values of the function $f(x)$ be represented by the ordinates to a curve. Its equation would then be

$$y = f(x).$$

It is proposed to find an expression for the area bounded by this curve, the x-axis, and two ordinates AP and BQ, corresponding to two given values of x, $x = a$ and $x = b$, respectively.

Let the interval from A to B be divided into n equal intervals AA_1, A_1A_2, $\cdot \cdot \cdot$, $A_{n-1}B$ each of magnitude Δx, so that

$$\text{interval } AB = b - a = n\Delta x.$$

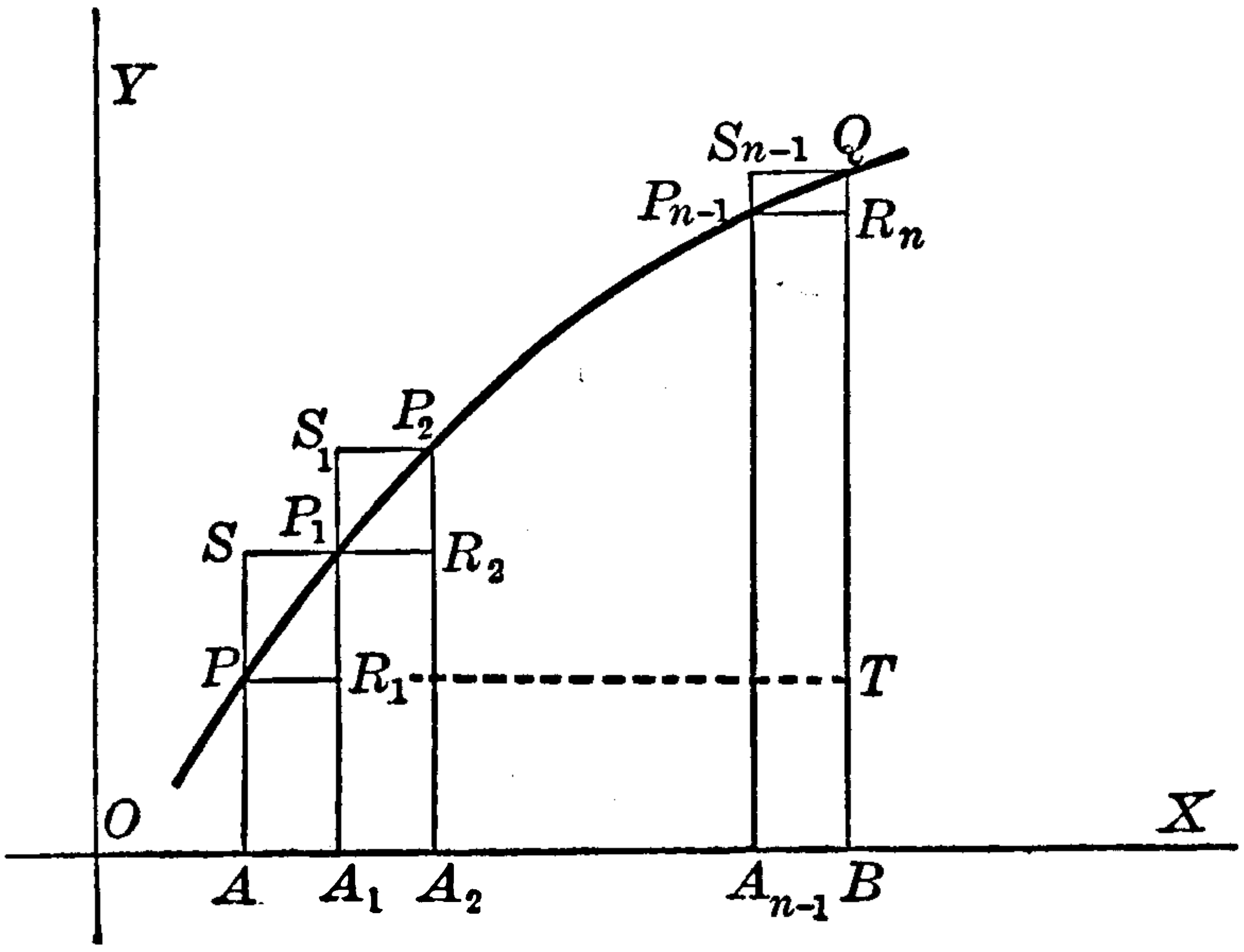

FIG. 61.

At each of the points of division A, A_1, $\cdot \cdot \cdot$, B erect ordinates, and suppose that these meet the curve in the points P, P_1, $\cdot \cdot \cdot$, Q. Through the latter points draw lines PR_1, P_1R_2, $\cdot \cdot \cdot$, $P_{n-1}R_n$ parallel to the x-axis.

A series of rectangles PA_1, P_1A_2, · · · is thus formed, each of which lies entirely within the given area. These will be referred to as the *interior* rectangles. By producing the lines already drawn, a series of rectangles SA_1, S_1A_2, · · · is formed which will be called the *exterior* rectangles. It is clear that the value of the given area will always lie between the sum of the interior, and the sum of the exterior rectangles, or, expressed in a formula,

$$PA_1 + P_1A_2 + \cdots + P_{n-1}B < \text{area } APQB < SA_1 + S_1A_2 + \cdots + S_{n-1}B. \tag{7}$$

The difference between the sum of the exterior and the sum of the interior rectangles is

$$SR_1 + S_1R_2 + \cdots + S_{n-1}R_n = \text{rectangle } S_{n-1}T = TQ \cdot \Delta x.$$

If the function $f(x)$ does not become infinite as x varies from a to b, TQ will be finite and hence $TQ \cdot \Delta x$ will approach zero simultaneously with Δx. Hence the limit of the sum of the exterior rectangles equals the limit of the sum of the interior rectangles. From (7) it follows that the area is equal to the common limit of these two sums.

To determine this sum observe that

$$\text{Rectangle } APR_1A_1 = AP \cdot AA_1 = f(a) \cdot \Delta x.$$

Similarly $$A_1P_1R_2A_2 = f(a + \Delta x) \cdot \Delta x,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_{n-1}P_{n-1}R_nB = f(a + \overline{n-1}\,\Delta x) \cdot \Delta x.$$

Adding,
sum of rectangles

$$= [f(a) + f(a + \Delta x) + \cdots + f(a + \overline{n-1}\,\Delta x)]\Delta x.$$

Hence, by requiring Δx to approach zero,

Area $APQB$

$$= \lim_{\Delta x \doteq 0} [f(a) + f(a + \Delta x) + \cdots + f(a + \overline{n-1}\,\Delta x)]\Delta x. \tag{8}$$

The expression just obtained for the area is identical with that occurring in the right-hand member of (5), and affords one of the simplest and most interesting of the geometrical interpretations of that formula. Thus

$$\text{Area} = \int_a^b f(x)dx = \int_a^b y dx. \tag{9}$$

150. Generalization of the area formula. Positive and negative area. Instead of taking the limit of the sum of the interior (or exterior) rectangles, a more general procedure would be to take a series of intermediate rectangles.

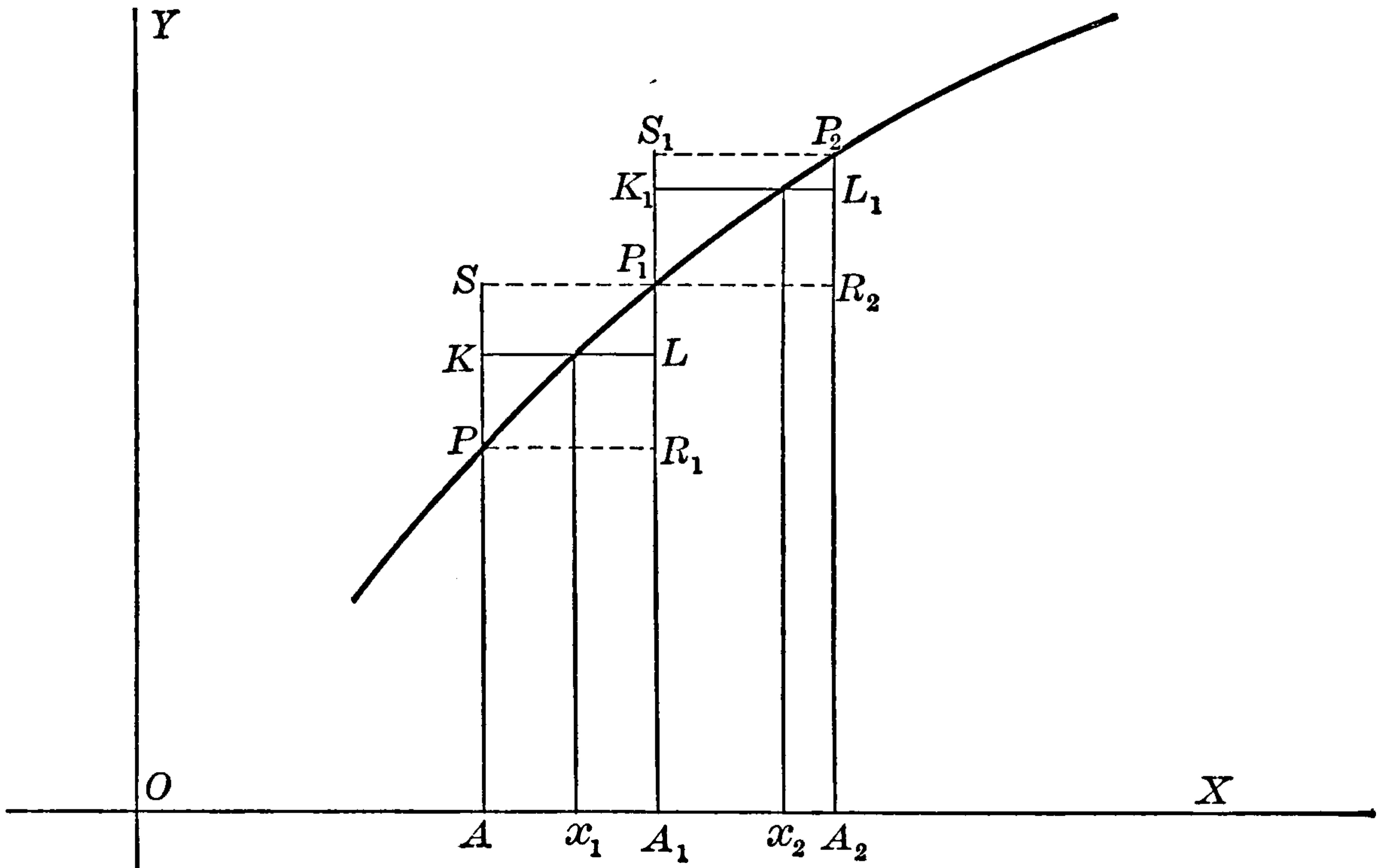

FIG. 62.

Let x_1 be any value of x between a and $a + \Delta x$, x_2 any value between $a + \Delta x$ and $a + 2\Delta x$, etc. Then $f(x_1)\Delta x$ would be the area of a rectangle KLA_1A (Fig. 62) intermediate between PA_1 and SA_1; that is,

$$PA_1 < f(x_1)\Delta x < SA_1.$$

Likewise $\quad P_1A_2 < f(x_2)\Delta x < S_1A_2, \quad$ etc.

Hence

Sum of interior rectangles $< [f(x_1)+f(x_2)+\cdots]\,\Delta x$

$<$ sum of exterior rectangles,

and therefore (cf. Fig. 61),

$$\text{Area } APQB = \lim_{\Delta x \doteq 0}[f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x. \quad (10)$$

If the area to be found is entirely above the x-axis, the ordinates are all positive. If at the same time Δx be taken positive (that is, if $b > a$), formula (8) or (10) gives a positive sign to the area. On the other hand, the area is negative if below the x-axis.

If the curve $y = f(x)$ is partly above and partly below the x-axis, the value of the definite integral (8) will be represented by the algebraic sum of the positive and negative areas limited by this curve.

151. Certain properties of definite integrals. From the definition of the definite integral $\int_a^b f(x)\,dx$ as the limit of a particular sum [formula (5), p. 250], certain important properties may be deduced.

(*a*) *Interchanging the limits a and b changes the sign of the definite integral.*

For if x starts at the upper limit b and diminishes by the addition of successive negative increments $(-\Delta x)$, a change of sign will occur in formula (5), giving

$$F(a) - F(b) = \int_b^a f(x)\,dx.$$

Hence
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \quad (11)$$

(*b*) *If c be a number between a and b $(a < c < b)$, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \quad (12)$$

(*c*) *The Mean Value Theorem.*

The area $APQB$ (Fig. 63), which represents the numerical value of the definite integral may be determined as follows:

Let an ordinate MN be drawn in such a position that

$$\text{area } PSN = \text{area } NRQ.$$

If ξ denote the value of x corresponding to the point N, then $MN = f(\xi)$, and

$$\text{Area } APQB = \text{rectangle } ASRB$$
$$= MN \cdot AB = f(\xi)(b-a).$$

Hence,

$$\int_a^b f(x)dx = f(\xi)(b-a), \qquad (13)$$

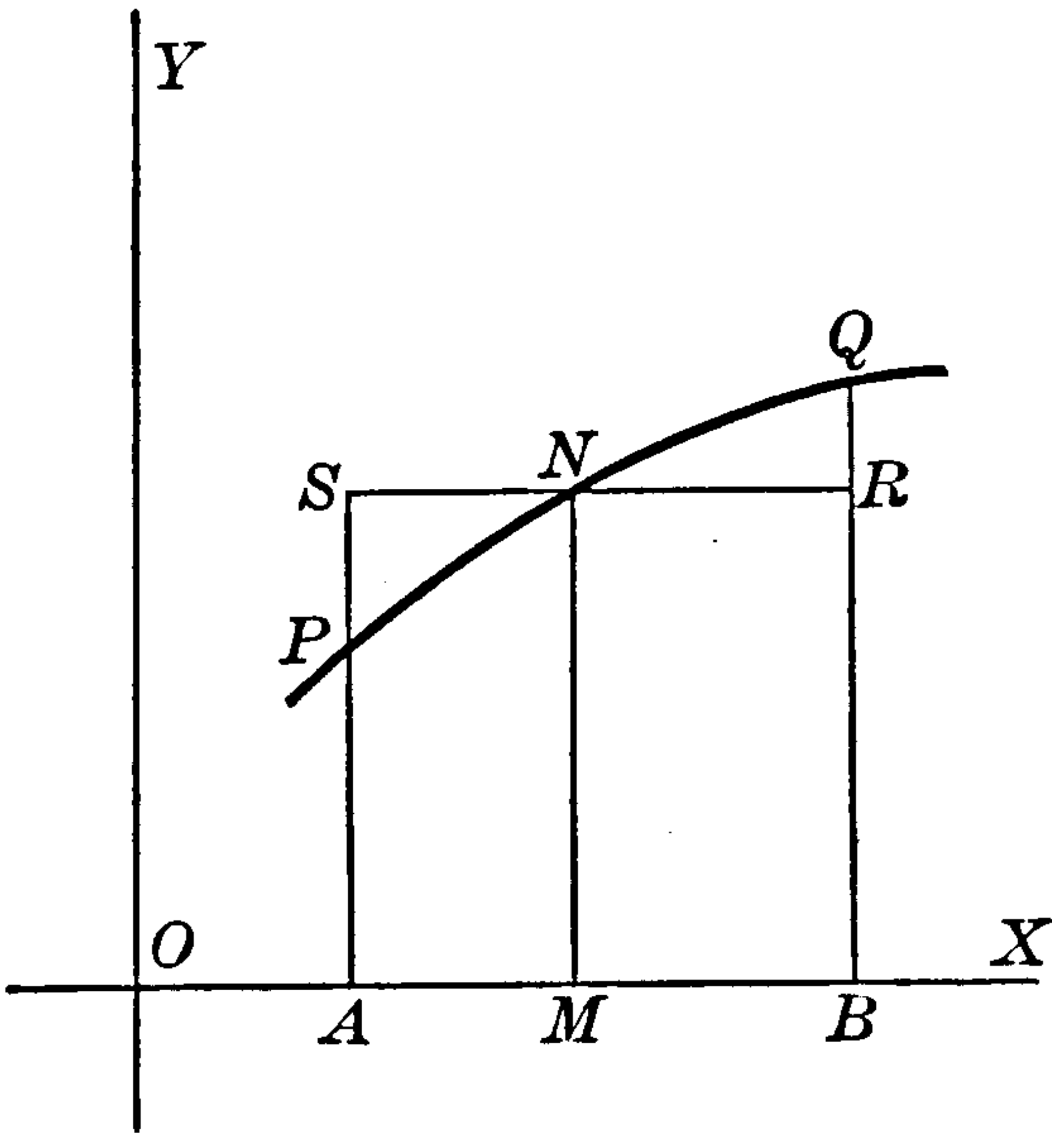

FIG. 63.

in which ξ is some value of x between a and b. This result is known as the Mean Value Theorem. (Compare Art. 45.)

The theorem may be expressed in words as follows:

The value of the definite integral

$$\int_a^b f(x)dx$$

is equal to the product of the difference between the limits by the value of the function $f(x)$ *corresponding to a certain value* $x = \xi$ *between the limits of integration.*

152. Definition of the definite integral when $f(x)$ becomes infinite. Infinite limits. In the preceding sections it has been assumed that $f(x)$ is always finite so long as x remains within the prescribed limits. It is now necessary to examine the cases in which $f(x)$ is infinite.

Suppose, in the first place, that $f(x)$ becomes infinite at the upper limit $x = b$, but is elsewhere finite. In that event, take for upper limit a value $x = x'$, which is less than b, $a < x' < b$. Then, according to the preceding results,

$$F(x') - F(a) = \int_a^{x'} f(x)dx.$$

Now let x' increase and approach b as limit. If at the same time the integral

$$\int_a^{x'} f(x)dx \tag{14}$$

approaches a definite, finite limit, that limit will be defined as the value of the integral

$$\int_a^b f(x)dx$$

in the case under consideration; that is,

$$\int_a^b f(x)dx = \lim_{x' \doteq b} \int_a^{x'} f(x)dx. \qquad x' < b.$$

On the other hand the integral (14) may increase without limit. When that happens, the integral will be said to have an infinite value, or

$$\int_a^b f(x)dx = \infty.$$

In a similar manner, if $f(a) = \infty$, the value of $\int_a^b f(x)dx$ will be defined to be the limit of the integral

$$\int_{x'}^b f(x)dx \qquad a < x' < b$$

as x' diminishes and approaches a as limit.

Finally, if $f(c) = \infty$, where c is any number between a and b, it is necessary to determine the meaning of

$$\int_a^c f(x)dx, \text{ and } \int_c^b f(x)dx$$

by the method just suggested, and then add the two results in accordance with formula (12).

Heretofore the limits a, b have been assumed to be finite. The case in which one of the limits, say b, is infinite, is readily disposed of by integrating from a to a finite upper limit x', and then considering the limit which the integral approaches as x' increases to infinity. This limit, when one exists, will be defined as the value of the integral, so that

$$\int_a^\infty f(x)dx = \lim_{x' \doteq \infty} \int_a^{x'} f(x)dx.$$

An exactly similar mode of procedure is to be followed if the lower limit a is $-\infty$, or if both limits are infinite.

EXERCISES

Ex. 1. Prove, without performing the integration, that

$$\int_{-a}^{+a} \frac{xe^{x^2}dx}{1+x^2} = 0.$$

$$\int_0^{-a} \frac{xe^{x^2}dx}{1+x^2} + \int_a^0 \frac{xe^{x^2}dx}{1+x^2}$$

$$-\cancel{\int_0^a \frac{xe^{x^2}dx}{1+x^2}} + \int_a^0 \frac{xe^{x^2}}{1}$$

Ex. 2. Without integrating show that

$$\int_{-a}^{2a} \frac{x\,dx}{x^2+a^2} = \int_a^{2a} \frac{x\,dx}{x^2+a^2}.$$

Ex. 3. If $y = \phi(x)$ and $y = \psi(x)$ are the equations of two curves which are continuous between $x = a$ and $x = b$, and such that to each value of x $(a < x < b)$ corresponds but one value of y, prove that the area bounded by these curves and the two ordinates $x = a$, $x = b$ is numerically equal to

$$\int_a^b [\phi(x) - \psi(x)]dx.$$

Ex. 4. Prove that the area of the circle $(x-h)^2 + (y-k)^2 = r^2$ is equal to

$$\int_{h-r}^{h+r} 2\sqrt{r^2 - (x-h)^2}dx.$$

Ex. 5. Evaluate $\displaystyle\int_0^\infty \frac{dx}{a^2+x^2}$.

Ex. 6. Evaluate $\displaystyle\int_1^2 \frac{dx}{\sqrt{x-1}}$.

Ex. 7. Evaluate $\displaystyle\int_0^2 \frac{dx}{(x-1)^{\frac{4}{3}}}$.

CHAPTER VII

GEOMETRICAL APPLICATIONS

153. Areas. Rectangular coördinates. It was shown in Art. 149 that the area bounded by the curve $y = f(x)$, the x-axis, and the two ordinates $x = a$, $x = b$, is represented by the definite integral

$$\int_a^b f(x)\,dx = \int_a^b y\,dx. \qquad (1)$$

In an exactly similar manner it can be shown that the area limited by the curve, the y-axis, and the two abscissas $y = \alpha$, $y = \beta$, is represented by

$$\int_\alpha^\beta x\,dy. \qquad (2)$$

It was remarked at the end of Art. 150 that when b is greater than a the integral (1) gives a positive or negative result according as the area is above or below the x-axis.

Similarly, if $\beta > \alpha$, the integral (2) gives a positive or negative result according as the area which it represents is to the right or left of the y-axis.

Whenever it is required to determine the area of a figure which is partly on one side and partly on the other side of the coördinate axis, it is necessary to calculate the positive and the negative areas separately and add the results, each taken with a positive sign. [Cf. Ex. 5, p. 262.]

154. Second method. Another method of determining the area is based on the result of Art. 10, p. 23. It was there shown that if z represents the area measured from a fixed ordinate AP (at $x = a$) up to an ordinate MN corresponding to a variable abscissa x, then the derivative of area with respect to x is equal to the function $f(x)$; that is

$$\frac{dz}{dx} = y = f(x),$$

or, in the differential notation,

$$dz = ydx = f(x)dx.$$

The area z may accordingly be found by integrating $f(x)$.

Hence $z = \int f(x)dx + C.$

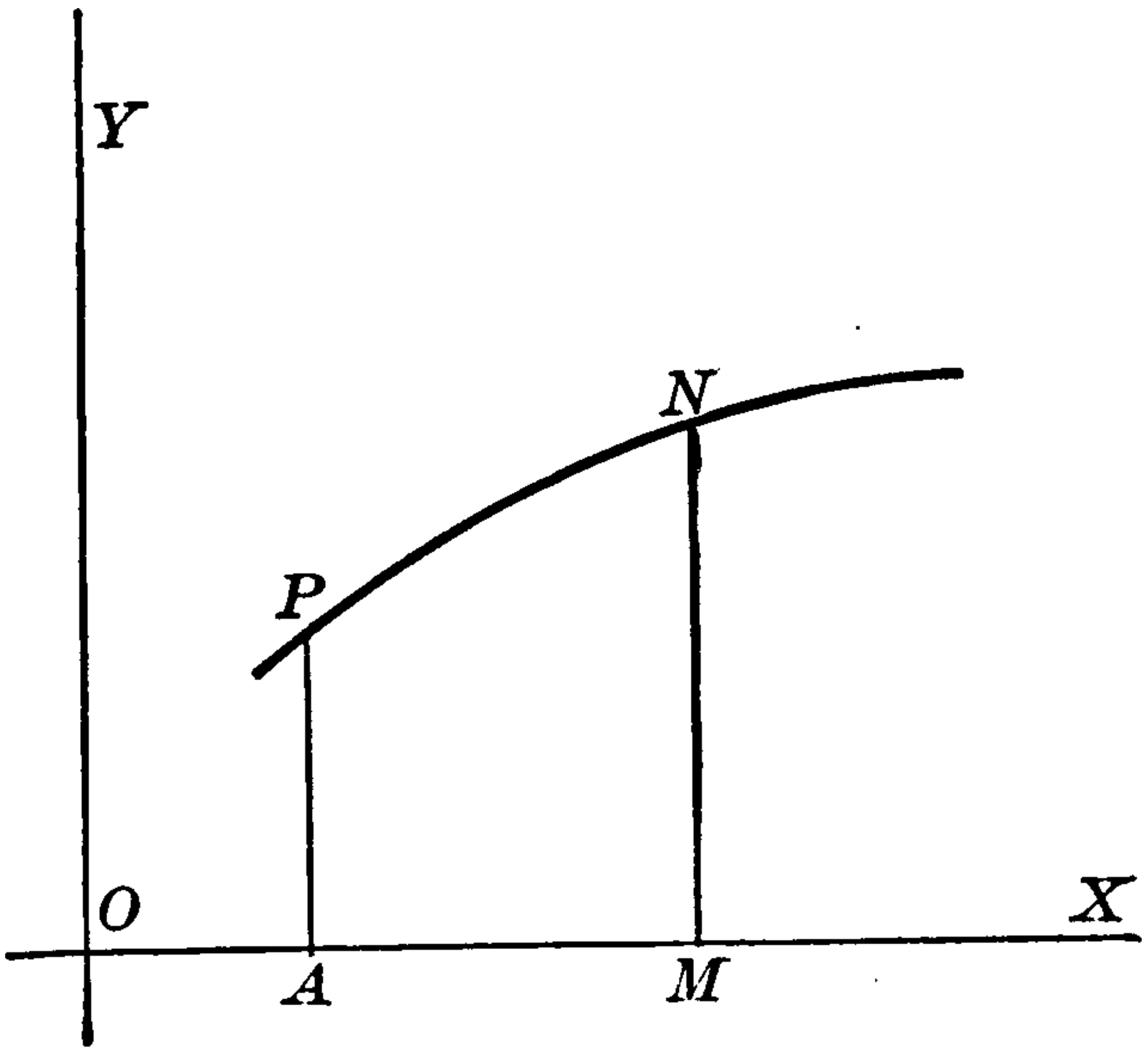

FIG. 64.

The value of the constant of integration C is determined by the condition that when $x = a$, z must be zero, since in that event the ordinate MN coincides with the initial position.

Ex. Find the area bounded by the curve $y = \log x$, the x-axis, and the two ordinates $x = 2$, $x = 3$.

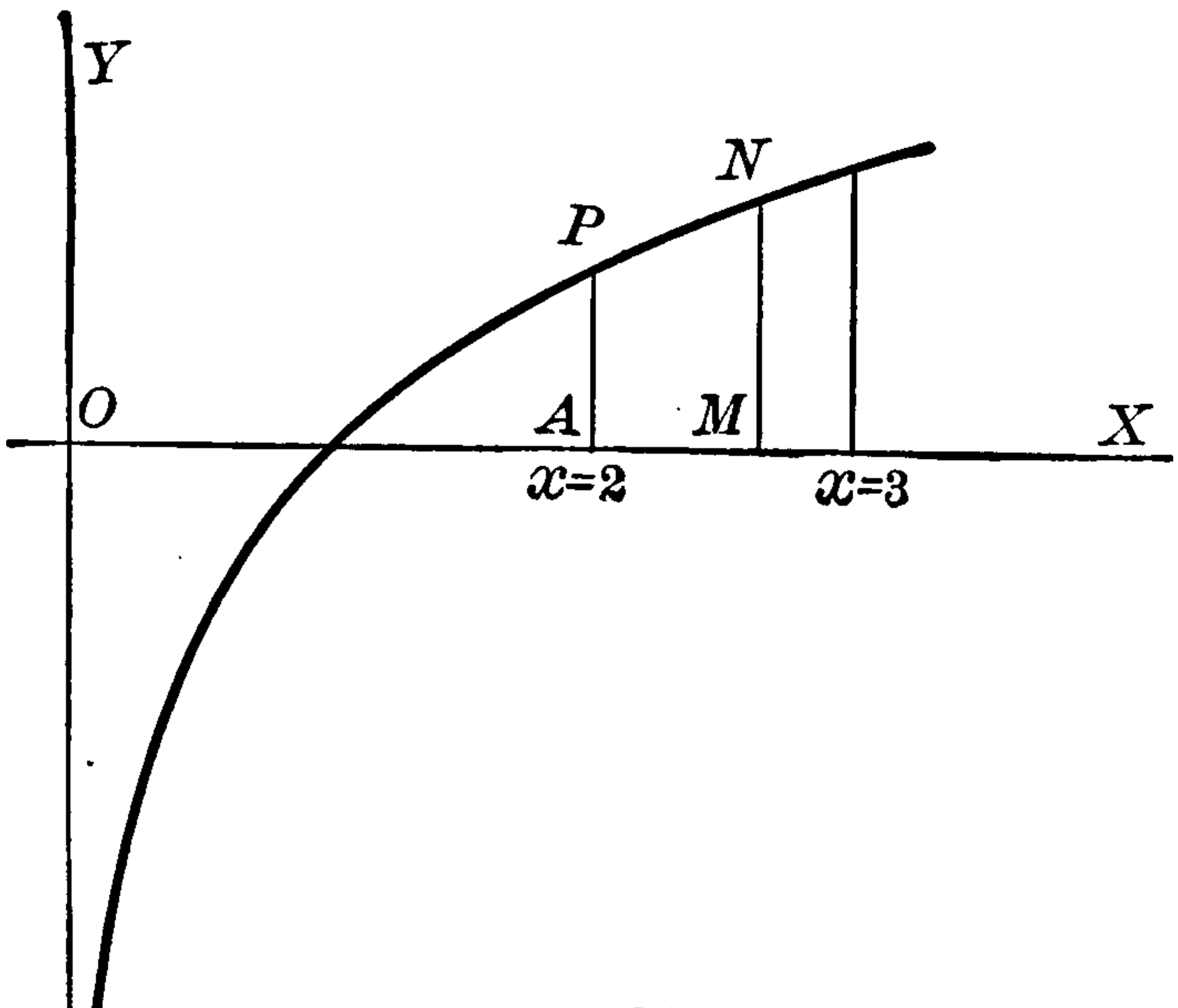

FIG. 65.

$$\text{Area } APNM = \int \log x \, dx + C$$
$$= x(\log x - 1) + C.$$

Since the area is zero when $x = 2$, it follows that

$$0 = 2 \log 2 - 2 + C,$$

whence

$$C = 2 - 2 \log 2.$$

Accordingly

$$x(\log x - 1) + 2 - 2 \log 2$$

represents the area measured from the ordinate $x = 2$ up to the variable ordinate MN. When $x = 3$ the required area is found to be $3(\log 3 - 1) + 2 - 2 \log 2 = \log \frac{27}{4} - 1.$

EXERCISES

1. Find the area bounded by the parabola $y = 4ax^2$, the x-axis, and the ordinate $x = b$.

2. Find the area of the triangle formed by the line $\frac{x}{a} + \frac{y}{b} = 2$ and the coördinate axes.

3. Find the area between the x-axis and one semi-undulation of the curve $y = \sin x$.

4. Find the area bounded by the semi-cubical parabola $y^2 = ax^3$ and the line $x = 5$.

5. Find the area between the curve $y = \sin^2 x \cos x$ and the x-axis, from the origin to the point at which $x = 2\pi$.

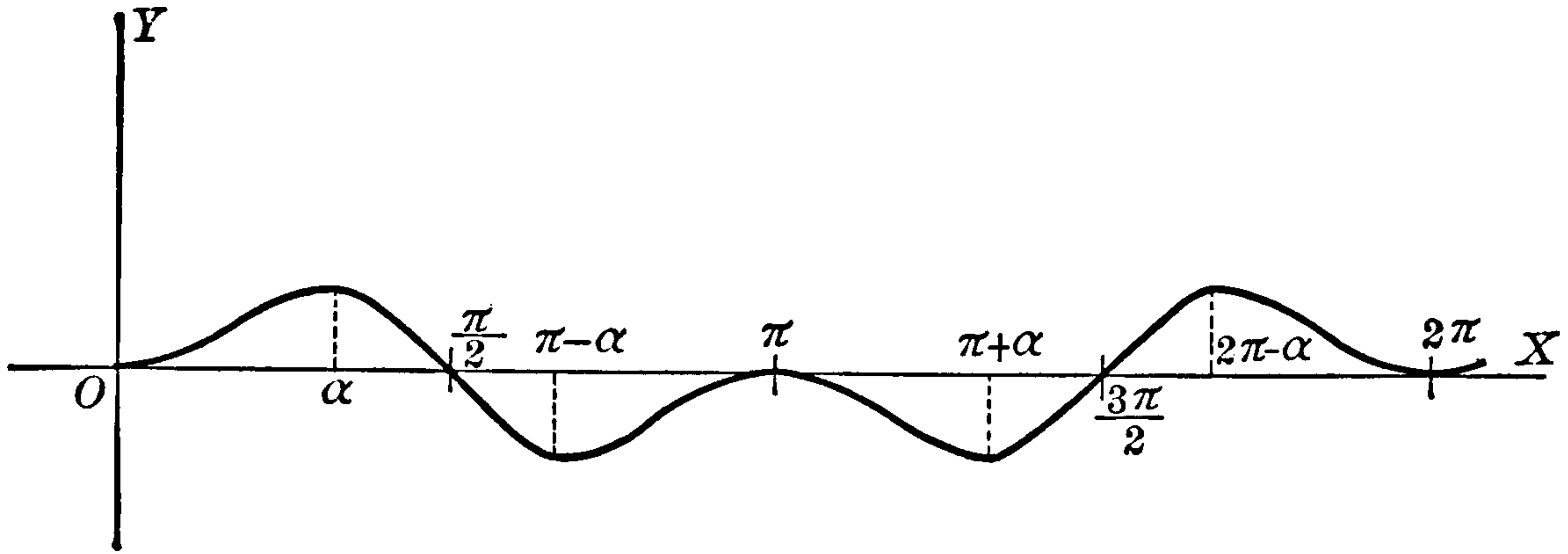

FIG. 66.

An examination of the curve will show that the area is partly above and partly below the x-axis. The curve crosses the axis at $x = \frac{\pi}{2}$, and at $x = \frac{3\pi}{2}$.

The first portion of area, which is positive, is obtained by integrating from 0 to $\frac{\pi}{2}$. The result is $\frac{1}{3}$. The next two portions of area are negative, and are calculated by integrating from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. The result is $-\frac{2}{3}$. The last portion, which is positive, is found, by integrating from $\frac{3\pi}{2}$ to 2π, to be $\frac{1}{3}$. Hence total area $= \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{4}{3}$.

6. Find the area between the x-axis and the curve $y = a \sin 4x$, from the origin to $x = \pi$.

7. Find the area bounded by the cubical parabola $y = x^3$, the y-axis, and the line $y = 8$.

8. Find the area bounded by the parabola $y = x^2$ and the line $y = x$. [Cf. Ex. 3, p. 259.]

9. Find the area bounded by the parabola $y = x^2$ and the two lines $y = x$, and $y = 2x$.

10. Find the area bounded by the parabola $y^2 = 4px$ and the line $x = a$, and show that it is two thirds the area of the circumscribing rectangle.

What is the area bounded by the curve and its latus rectum?

11. Find the area of the circle $x^2 + y^2 + 2ax = 0$.

12. Find the area bounded by the coördinate axes, the witch $y = \frac{8a^3}{x^2 + 4a^2}$, and the ordinate $x = x_1$. By increasing x_1 without limit, find the area between the curve and the x-axis.

13. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

14. Find the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

15. Find the area bounded by the logarithmic curve $y = a^x$, the x-axis, and the two ordinates $x = x_1$, $x = x_2$. Show that the result is proportional to the difference between the ordinates.

155. Precautions to be observed in evaluating definite integrals. The two methods just given for determining plane areas are essentially alike in the processes required, namely:

(1) to find the integral of the given function $f(x)$;

(2) to substitute for x the two limiting values a and b, and subtract the first result from the second.

Erroneous results may be reached, however, by an incautious application of this process.

In practical problems, the case requiring special care is that in which $f(x)$ becomes infinite for some value of x between a and b. When that happens, a special investigation must be made after the manner of Art. 152.

Ex. 1. Find the area bounded by the curve $y(x-1)^2 = c$, the coördinate axes, and the ordinate $x = 2$.

A direct application of the formula gives

$$\text{area} = \int_0^2 \frac{c\,dx}{(x-1)^2} = -\frac{c}{x-1}\Big]_0^2 = -2c,$$

where the symbol $\Big]_a^b$ is a sign of substitution, indicating that the values b, a are to be inserted for x in the expression immediately preceding the sign, and the second result subtracted from the first.

This result is incorrect. A glance at the equation of the curve shows that $f(x)\left[= \frac{c}{(x-1)^2}\right]$ becomes infinite for $x = 1$. It is accordingly necessary to find the area *OCPA* (Fig. 67) bounded by an ordinate *AP* corresponding to a value $x = x'$ which is less than 1. For this portion the area $f(x)$ is finite and positive, and formula (1) can be immediately applied, with the result

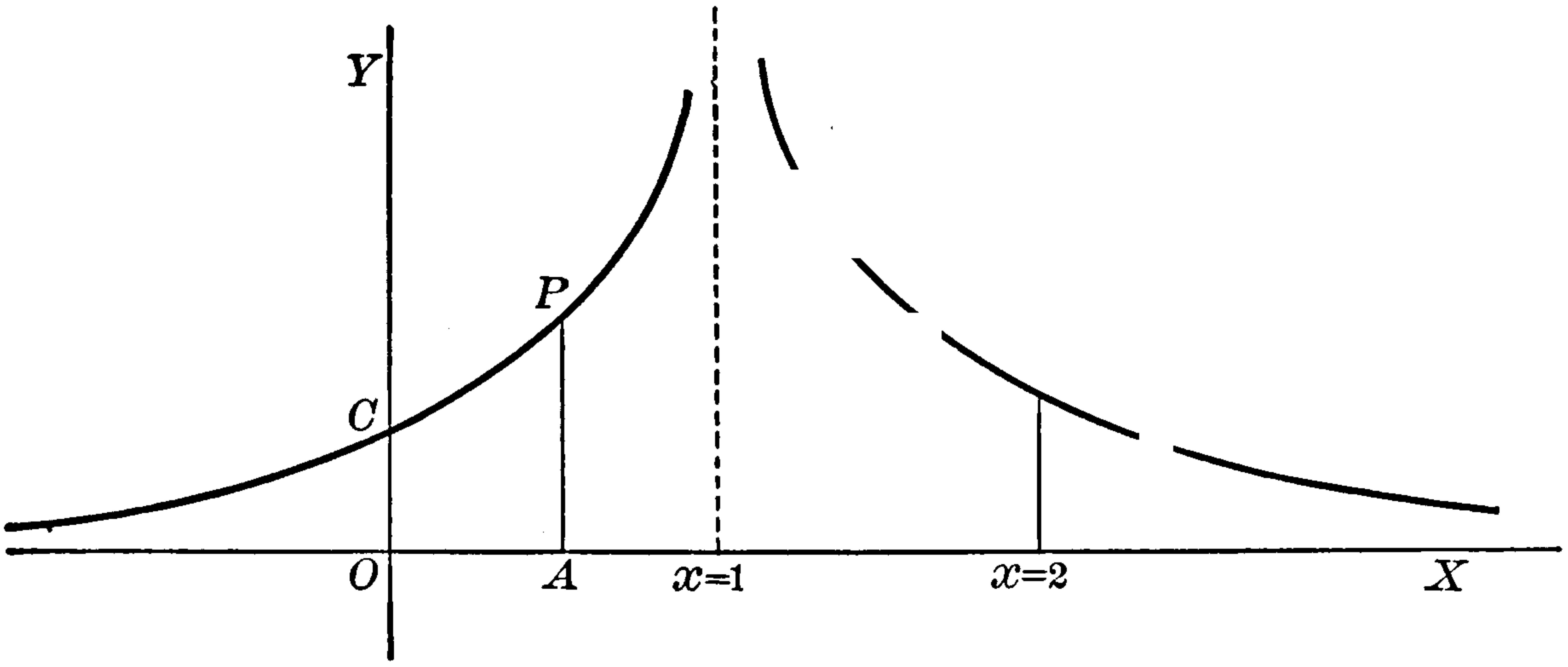

FIG. 67.

$$\text{area } OCPA = \int_0^{x'} \frac{c\,dx}{(x-1)^2} = -\frac{c}{(x-1)}\Big]_0^{x'} = -\frac{c}{x'-1} - c. \quad 0 < x' < 1.$$

If now x' be made to increase and approach 1 as a limit, the value of the expression for the area will increase without limit.

A like result is obtained for the area included between the ordinates $x = 1$ and $x = 2$. Hence the required area is infinite.

Ex. 2. Find the area limited by the curve $y^3(x^2 - a^2)^2 = 8x^3$, the coördinate axes, and the ordinate $x = 3a$.

Since $f(x)\left[=\dfrac{2x}{(x^2-a^2)^{\frac{2}{3}}}\right]$ becomes infinite for $x=a$, it is necessary in the first place to consider the area OPA (Fig. 68) and determine what

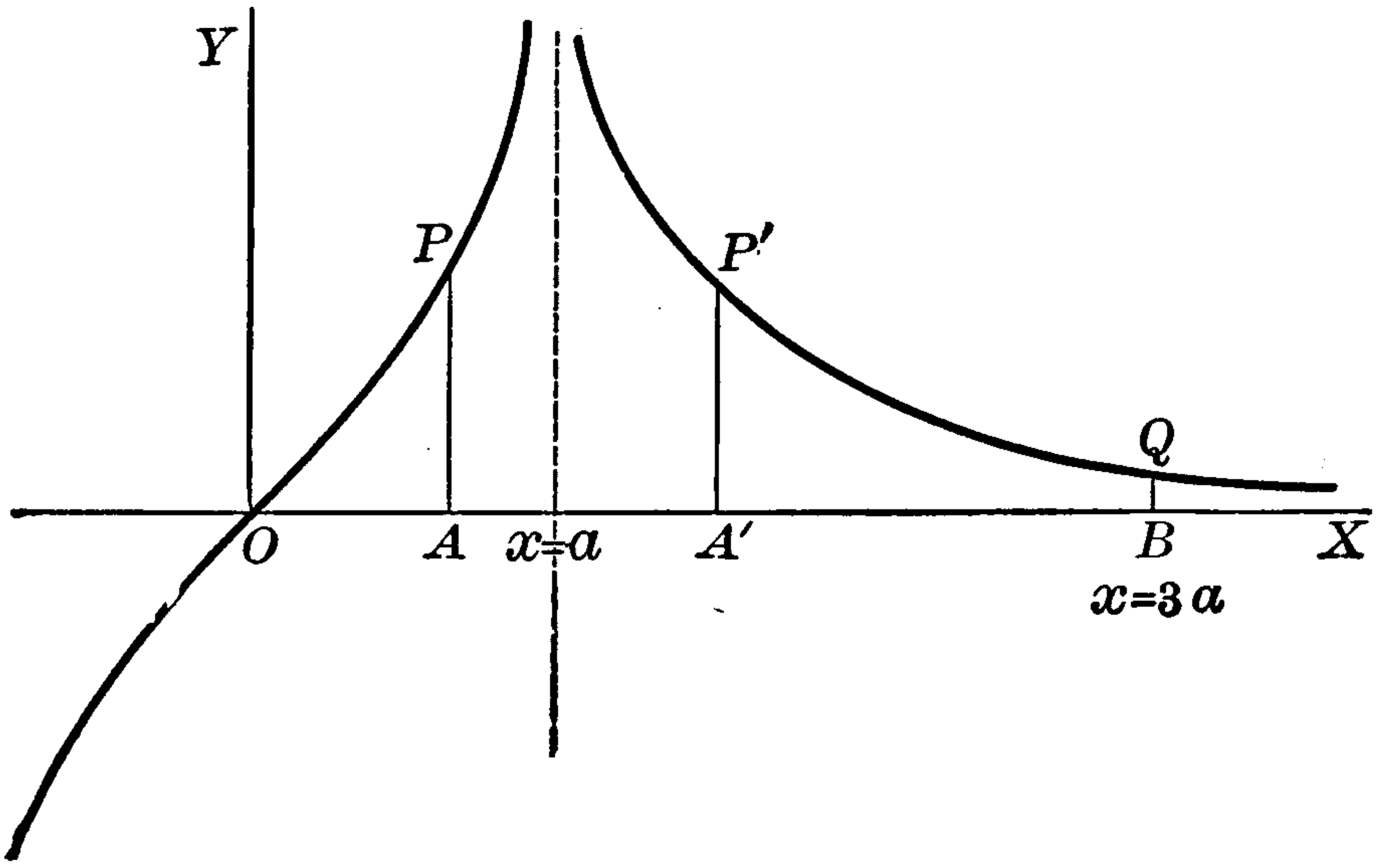

FIG. 68.

limit it approaches as AP approaches coincidence with the ordinate $x=a$. Accordingly

$$\text{area } OPA = \int_0^{x'} \frac{2x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 3(x^2-a^2)^{\frac{1}{3}}\Big]_0^{x'}$$

$$= 3(x'^2-a^2)^{\frac{1}{3}} + 3a^{\frac{2}{3}}, \qquad 0 < x' < a.$$

Whence

$$\lim_{x' \doteq a}[\text{area } OPA] = 3a^{\frac{2}{3}}.$$

In the same manner, the area $A'P'QB$ has the value

$$\int_{x'}^{3a} \frac{2x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 6a^{\frac{2}{3}} - 3(x'^2-a^2)^{\frac{1}{3}}, \qquad a < x' < 3a.$$

As x' diminishes towards a, the area increases to the limiting value $6a^{\frac{2}{3}}$. Hence, by adding the two results, the required area is found to be

$$3a^{\frac{2}{3}} + 6a^{\frac{2}{3}} = 9a^{\frac{2}{3}}.$$

The same result is found by a direct application of (1), viz.:

$$\int_0^{3a} \frac{2x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 3(x^2-a^2)^{\frac{1}{3}}\Big]_0^{3a} = 9a^{\frac{2}{3}},$$

so that in this case an immediate use of the area-formula gives the correct result.

Ex. 3. Find the area bounded by curve $y = \tan^{-1}x$, the coördinate axes, and the line $x = 1$.

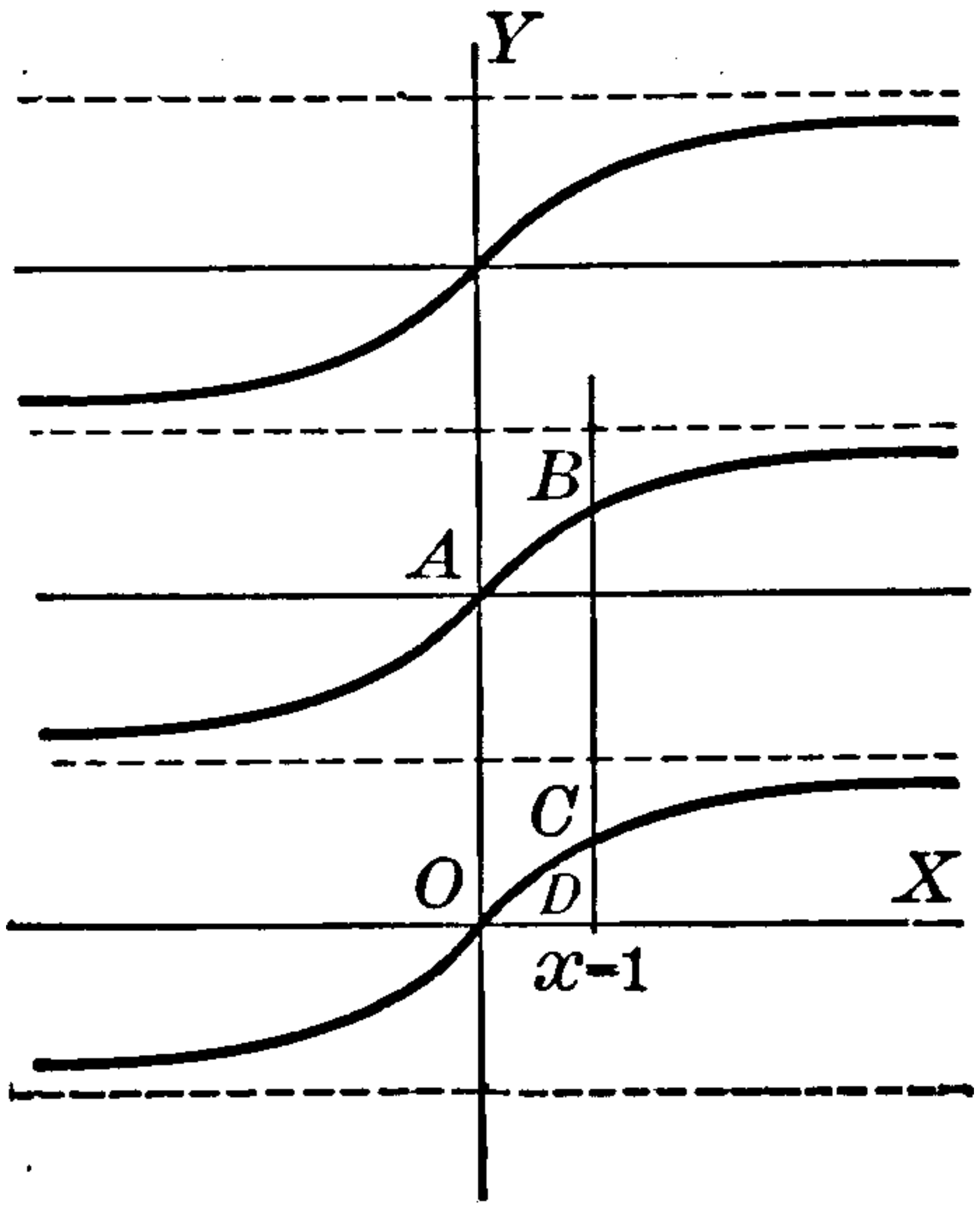

Fig. 69.

In this problem we have to deal with a many-valued function of x. In fact, to each value of x corresponds an infinite number of values of $\tan^{-1}x$. The problem accordingly has an indefiniteness which must be removed by making some additional assumption.

The curve $y = \tan^{-1}x$ consists of an infinite number of branches, corresponding ordinates of which differ by integer multiples of π. Each branch is continuous for all finite values of x (see Fig. 69). It is evidently necessary to select one of these branches for the boundary of the proposed area, and discard all the others. Suppose, for example, the branch AB is selected. The ordinate to this branch has the value π when x is zero, and increases continuously to $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ as x increases continuously to 1. Hence the required area is

$$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x - \tfrac{1}{2}\log(x^2+1)\right]_0^1$$

$$= \frac{5\pi}{4} - \tfrac{1}{2}\log 2.$$

Ex. 4. Find the area of the parallelogram strip $ABCO$ (Fig. 69).

Ex. 5. Find the area between the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and its asymptote $x = 2a$.

Ex. 6. Find the area inclosed by the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Ex. 7. Find the area bounded by the curve $a^2x = y(x - a)$, the x-axis, and the asymptote $x = a$.

Ex. 8. Find the area included between the curve $(2 - x)y^2 = x^2(x - 1)^2$ and its asymptote.

Ex. 9. What restriction must be placed on the exponent k in order that the area bounded by the curve $(1 - x)^k y = 1$, its asymptote $x = 1$, and the coördinate axes may be finite?

156. Areas. Polar Coördinates. Let PQ be an arc of a curve whose equation in polar coördinates is

$$\rho = f(\theta). \tag{3}$$

Let it be required to find the area bounded by this curve and the two radii OP and OQ.

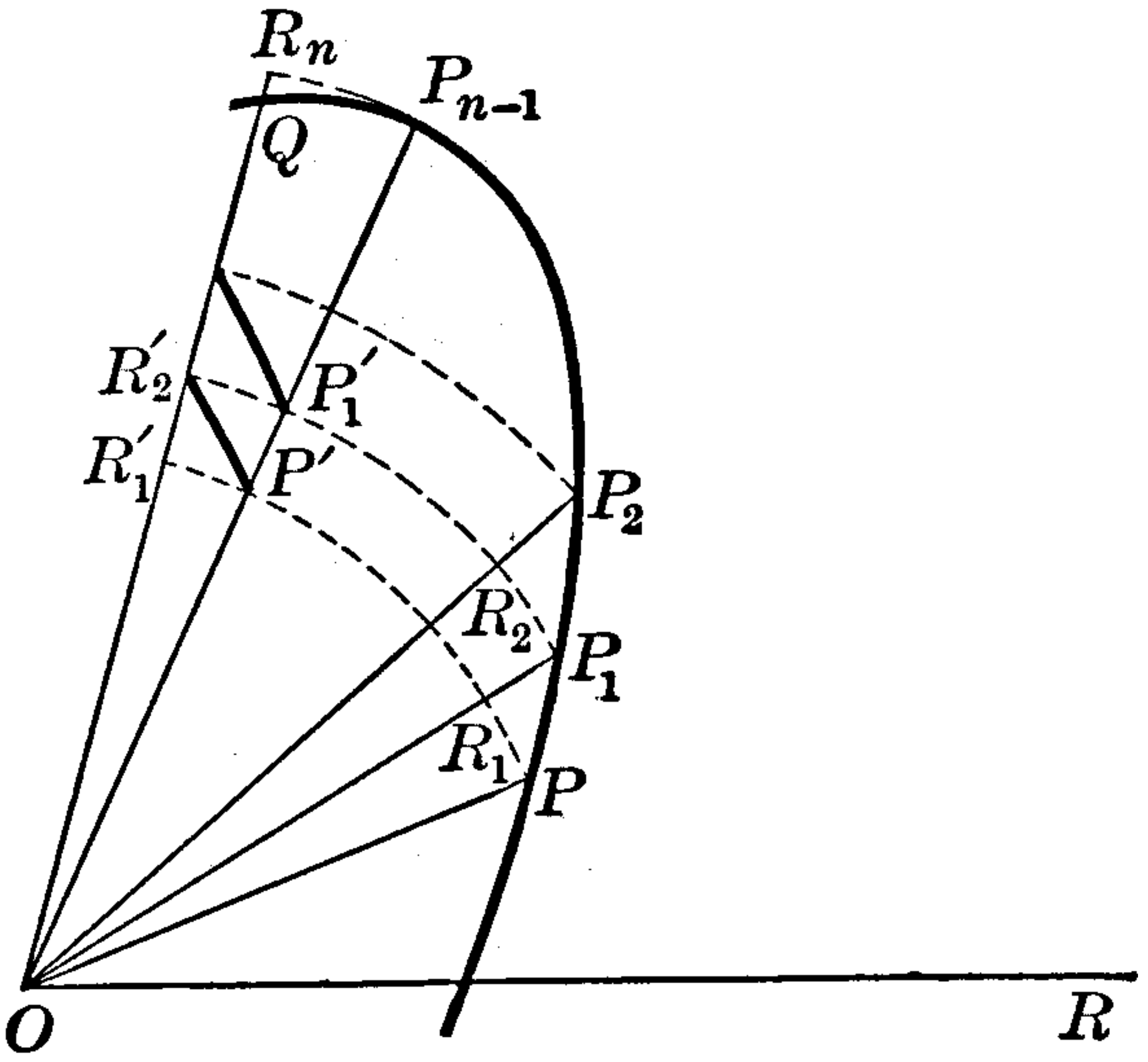

FIG. 70.

Draw from the origin a series of radii OP_1, OP_2, $\cdots$, OP_{n-1} at equal angles $\Delta\theta$. Let the coördinates of the points P, P_1, P_2, $\cdots$, Q be (a, α), (ρ_1, θ_1), (ρ_2, θ_2), $\cdots$, (b, β). Draw the circle arcs PR_1R_1', $P_1R_2R_2'$, $\cdots$. In the circular sector POR_1,

$$\text{radius } OP = a,$$

$$\text{arc } PR_1 = a \cdot \Delta\theta;$$

hence

$$\text{area } POR_1 = \tfrac{1}{2} a^2 \Delta\theta.$$

Similarly

$$\text{area } P_1OR_2 = \tfrac{1}{2} \rho_1{}^2 \Delta\theta,$$

$$\text{area } P_2OR_3 = \tfrac{1}{2} \rho_2{}^2 \Delta\theta,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\text{area } P_{n-1}OR_n = \tfrac{1}{2} \rho_{n-1}{}^2 \Delta\theta.$$

The sum of these sectorial areas is

$$\tfrac{1}{2}(a^2 + \rho_1{}^2 + \rho_2{}^2 + \cdots + \rho_{n-1}{}^2)\,\Delta\theta. \tag{4}$$

This is an approximate value for the required area POQ, which is less than the true value by the amounts contained in the neglected triangular portions PR_1P_1, $P_1R_2P_2$, etc. Suppose the figure PR_1P_1 revolved about O until it occupies the position $P'R_1'R_2'$, and similarly with $P_1R_2P_2$, etc. Then the sum of all the parts neglected is evidently less than the strip $P'R_1'R_nP_{n-1}$, the area of which approaches zero as the sectorial angle $\Delta\theta$ is made to approach zero.

Hence
$$\text{area } POQ = \lim_{\Delta x \doteq 0} \tfrac{1}{2}(a^2 + \rho_1^2 + \rho_2^2 + \cdots + \rho_{n-1}^2)\Delta\theta$$
$$= \int_\alpha^\beta \tfrac{1}{2}\rho^2\, d\theta.$$

Another method of procedure is illustrated in Ex. 1 immediately following.

Ex. 1. Find the area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.

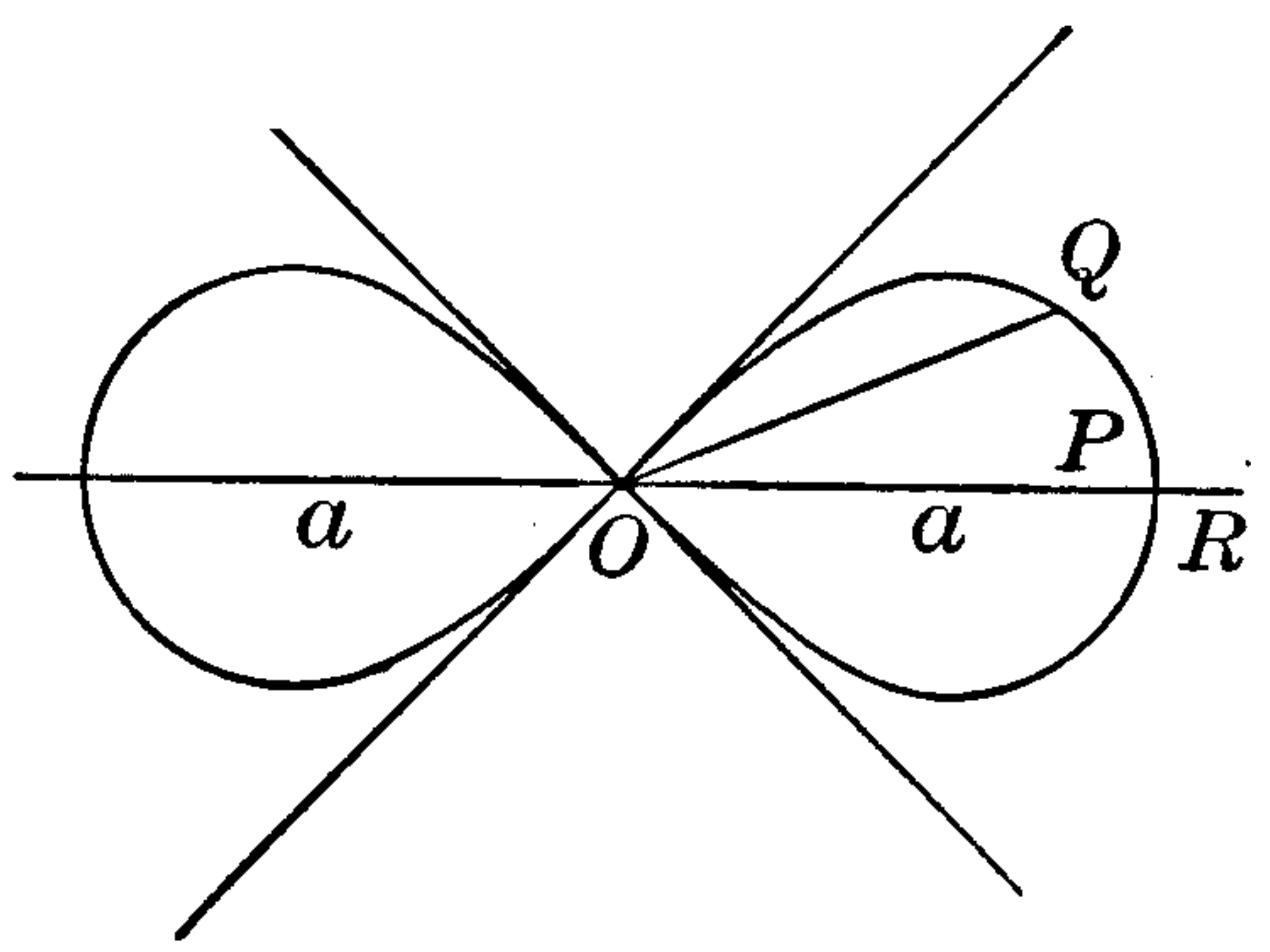

FIG. 71.

Let A denote the area of the sector POQ measured from the polar axis to an arbitrary radius vector OQ. The differential of area is (Art. 88, p. 142)

$$dA = \tfrac{1}{2}\rho^2 d\theta = \tfrac{1}{2}a^2 \cos 2\theta\, d\theta,$$

whence, by integration,

$$A = \tfrac{1}{2}a^2 \int \cos 2\theta\, d\theta$$
$$= \frac{a^2}{4}\sin 2\theta + C.$$

If θ were zero, the line OQ would occupy the initial position OP, and the area would be zero. That is

$$A = 0 \text{ when } \theta = 0.$$

The substitution of this result in the preceding formula gives

$$0 = 0 + C.$$

Hence
$$C = 0,$$
and
$$A = \frac{a^2}{4}\sin 2\theta.$$

In order to find the total area of the figure put $\theta = \frac{\pi}{4}$. In this case OQ will be tangent to the lemniscate at O. On account of the symmetry of the curve, the result obtained will be one fourth the total area, and therefore

$$\text{area} = a^2.$$

Ex. 2. Find the area of the cardioid $\rho = a(1 - \cos\theta)$.

Ex. 3. Find the area of the three loops of the curve $\rho = a \sin 3\theta$.

Ex. **4.** Find the area bounded by the hyperbolic spiral $\rho\theta = a$ and the two radii ρ_1, ρ_2. Show that the area is proportional to the difference between the radii.

Ex. **5.** Find the area limited by the parabola $\rho = a\sec^2\frac{\theta}{2}$ and its latus rectum. ✓

Ex. **6.** Find the area of the circle $\rho = 2a\cos\theta$.

Ex. **7.** Find the area of the four loops of the curve $\rho = a\sin 2\theta$. ✓

Ex. **8.** If $AB - H^2$ is positive, the equation $Ax^2 + 2Hxy + By^2 = 1$ represents an ellipse. By transforming to polar coördinates find its area.

157. Length of curves. Rectangular coördinates. Let it be required to determine the length of a continuous arc PQ of a curve whose equation is written in rectangular coördinates (x, y).

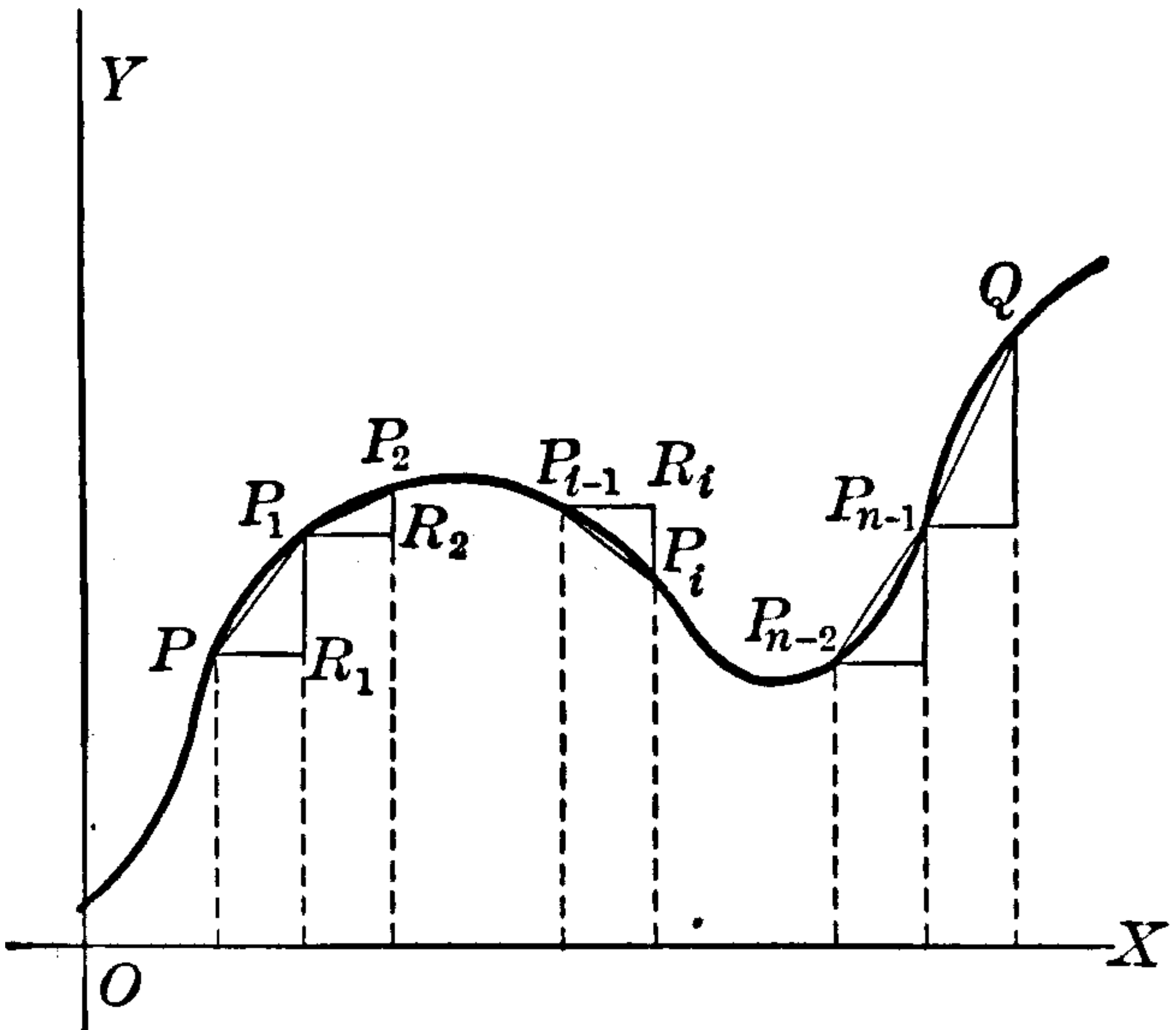

FIG. 72.

It is first necessary to define what is meant by the length of a curve. For this purpose, suppose a series of points P_1, P_2, ⋯, P_{n-1} taken on the arc PQ (Fig. 72), and imagine the lengths of the chords PP_1, P_1P_2, ⋯ to have been determined. The limit of the sum of these chords as the length of each chord approaches the limit zero will be defined, in accordance with accepted usage, as the length of the arc PQ;* that is,

$$\text{arc } PQ = \text{Lt}\,(\text{chord } PP_1 + \text{chord } P_1P_2 + \cdots + \text{chord } P_{n-1}Q). \quad (5)$$

---

* That this limit is always the same no matter how the points P_i are chosen, so long as the curve has a continuously turning tangent, and the distances $P_{i-1}P_i$ are all made to tend towards zero, admits of rigorous proof. The proof is, however, unsuitable for an elementary text-book. [See "Rouché et Comberousse, Traité de géométrie." Paris, 1891, part I, p. 189.]

This definition is immediately convertible into a formula suitable for direct application.

For, let the points P_1, P_2, $\cdots$ be so chosen that

$$PR_1 = P_1R_2 = \cdots = \Delta x,$$

the lines PR_1, etc. being drawn parallel to the x-axis. Denote by Δy the increment R_iP_i of y. Then the chord $P_{i-1}P_i$ has the length

$$P_{i-1}P_i = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$

$$= \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2}\Delta y.$$

It is clear that $\frac{\Delta y}{\Delta x}$ is the value of $\frac{dy}{dx}$ corresponding to some point of the curve between P_{i-1} and P_i. [Cf. Art. 45.] Hence, by substituting in (5) and using the principle employed in deriving the area-formula (10), Art. 150,

$$\text{arc } PQ = \int_{x'}^{x''} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{y'}^{y''} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \quad (6)$$

in which (x', y') and (x'', y'') are the coördinates of P and Q respectively.

The same result would also be obtained by integrating the expressions for the derivative of arc, (1) and (2), p. 139.

Ex. 1. Find the length of arc of the parabola $y^2 = 4px$ measured from the vertex to one extremity of the latus rectum.

In this case $$\frac{dy}{dx} = \sqrt{\frac{p}{x}},$$

and hence $$\text{length of arc} = \int_0^p \sqrt{1 + \frac{p}{x}}\, dx.$$

Ex. 2. Find the length of arc of the semi-cubical parabola $ay^2 = x^3$ from the origin to the point whose abscissa is $\frac{a}{4}$.

Ex. 3. Find the entire length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 4. Find the length of arc of the circle $(x-h)^2 + (y-k)^2 = r^2$.

Ex. 5. Find the length of arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from the vertex to the point (x_1, y_1).

Ex. 6. Find the length of the logarithmic curve $y = \log x$ from $x = 1$ to $x = \sqrt{3}$.

Ex. 7. Find the length of arc of the evolute of the ellipse

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

158. Length of curves. Polar coördinates. When the equation of the curve is given in polar coördinates, let the points $P_1, P_2, \cdots, P_{n-1}$ be so chosen that the vectorial angle θ increases by equal increments $\Delta\theta$ in passing from a point P_i on the curve to the next succeeding point P_{i+1}. Draw the lines P_iR_{i+1} perpendicular to the radii OP_{i+1}. Then

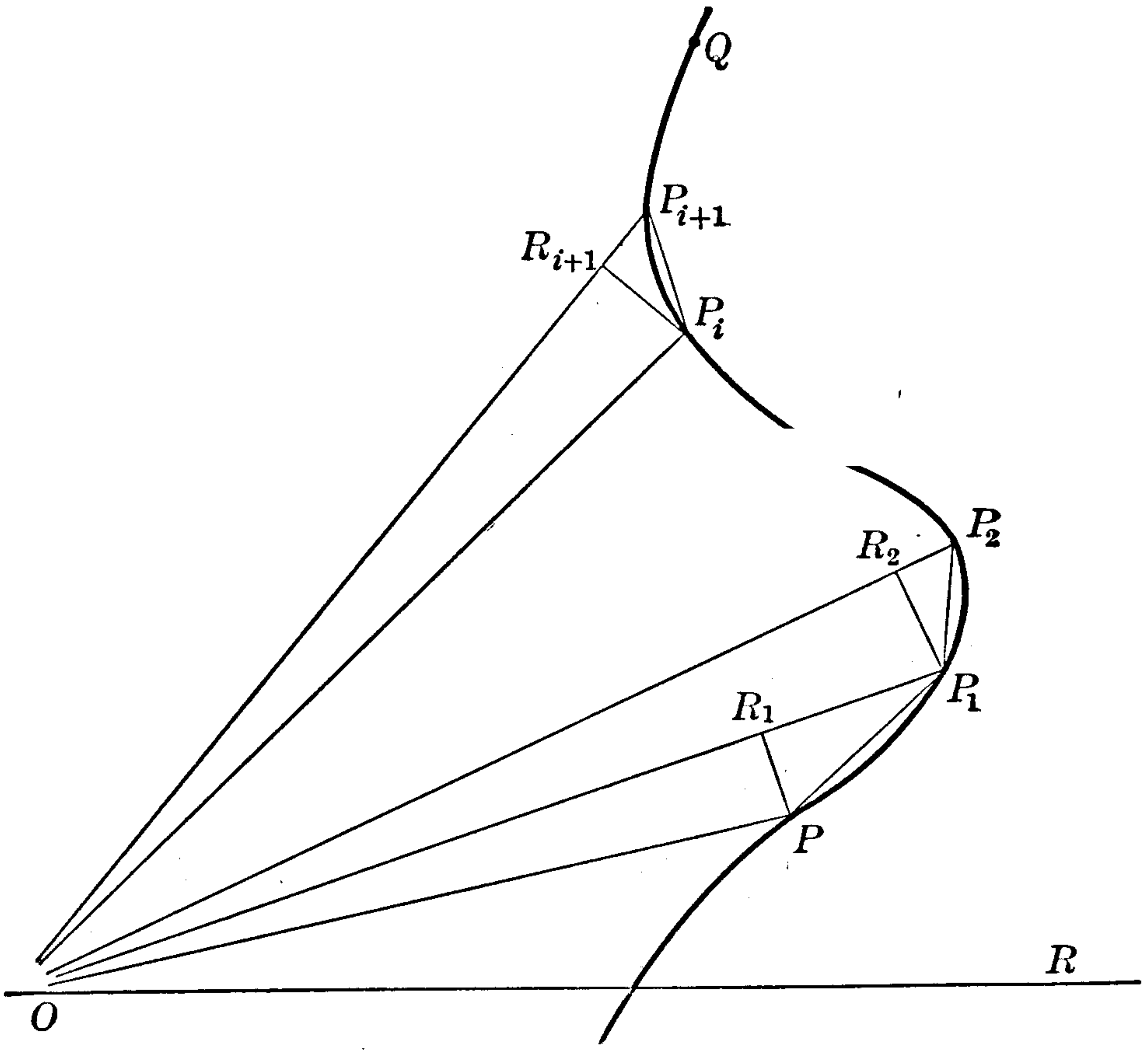

FIG. 73.

$$\text{chord } P_iP_{i+1} = \sqrt{\overline{P_iR_{i+1}}^2 + \overline{R_{i+1}P_{i+1}}^2} \qquad \text{[Cf. p. 141.]}$$

$$= \sqrt{(\rho \sin \Delta\theta)^2 + (\rho + \Delta\rho - \rho \cos \Delta\theta)^2}$$

$$= \Delta\rho \sqrt{\rho^2\left(\frac{\Delta\theta}{\Delta\rho}\right)^2\left(\frac{\sin \Delta\theta}{\Delta\theta}\right)^2 + \left[\rho\,\frac{\Delta\theta}{2}\cdot\frac{\Delta\theta}{\Delta\rho}\cdot\left(\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2 + 1\right]^2}$$

$$= \Delta\theta \sqrt{\rho^2\left(\frac{\sin \Delta\theta}{\Delta\theta}\right)^2 + \left[\frac{\Delta\rho}{\Delta\theta} + \rho\,\sin\tfrac{1}{2}\Delta\theta \cdot \frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right]^2}.$$

The limit of the sum of all such chords will be, according to definition, the length of the arc PQ. Hence

$$\text{arc } PQ = \int_{\rho'}^{\rho''} \sqrt{\rho^2\left(\frac{d\theta}{d\rho}\right)^2 + 1}\, d\rho = \int_{\theta'}^{\theta''} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta, \qquad (7)$$

in which (ρ', θ'), (ρ'', θ'') are the coördinates of P and Q respectively.

Ex. 1. Find the length of arc of the logarithmic spiral $\rho = e^{a\theta}$ between the two points (ρ_1, θ_1) and (ρ_2, θ_2).

Since $$\frac{d\rho}{d\theta} = ae^{a\theta},$$

it follows that $$\rho\,\frac{d\theta}{d\rho} = \frac{1}{a},$$

and $$\text{length of arc} = \int_{\rho_1}^{\rho_2} \sqrt{\frac{1}{a^2} + 1}\, d\rho = \sqrt{a^{-2} + 1}\,(\rho_2 - \rho_1).$$

Ex. 2. Find the length of arc of the cardioid $\rho = a(1 - \cos\theta)$.

Ex. 3. Find the length of the cissoid $\rho = 2a \tan\theta \sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

Ex. 4. Find the entire length of the curve $\rho = 2a \sin\theta$.

Ex. 5. Find the length of the parabola $\rho = a \sec^2 \frac{\theta}{2}$ between the points (ρ_1, θ_1) and (ρ_2, θ_2).

Ex. 6. Find the length of arc of the hyperbolic spiral $\rho\theta = a$ between the points (ρ_1, θ_1) and (ρ_2, θ_2).

159. Measurement of arcs by the aid of parametric representation. When the coördinates of a point on a given curve can be conveniently expressed in terms of a variable parameter, the problem of calculating its length is often simplified.

Suppose a curve has its rectangular coördinates expressible in the form

$$x = \phi(t), \quad y = \psi(t),$$

in which $\phi(t)$, $\psi(t)$ are single-valued functions of the variable t. Then

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}, \quad dx = \frac{dx}{dt}dt,$$

and formula (6), p. 270, becomes

$$\int_{t'}^{t''} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt, \tag{8}$$

in which t', t'' are the values of t corresponding to the extremities of the arc whose length is to be found.

In like manner, if (ρ, θ) are expressible in terms of a third variable t, formula (7), p. 272, becomes

$$\int_{t'}^{t''} \sqrt{\left(\rho \frac{d\theta}{dt}\right)^2 + \left(\frac{d\rho}{dt}\right)^2}\, dt.$$

✓ **Ex. 1.** Find the length of a complete arch of the cycloid

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

✓ **Ex. 2.** Find the length of the epicycloid

$$x = a(m\cos t - \cos mt), \quad y = a(m\sin t - \sin mt)$$

from $t = 0$ to $t = \dfrac{2\pi}{m-1}$.

✓ **Ex. 3.** Find the length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ by expressing x and y in the form $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Ex. 4. Find the length of the involute of the circle

$$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$$

from $\theta = 0$ to $\theta = \theta_1$.

Ex. 5. Find the length of arc of the curve $x^{\frac{2}{3}} - y^{\frac{2}{3}} = a^{\frac{2}{3}}$, from $(a, 0)$ to (x_1, y_1).

Assume $$x = a\sec^3\theta,\ y = a\tan^3\theta.$$

Ex. 6. Find the length of arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Putting $$x = a\cos\phi,\ y = b\sin\phi,$$

$$\text{complete arc} = \int_0^{2\pi}\sqrt{1 - e^2\cos^2\phi}\,d\phi$$
$$= 2\,a\pi\left[1 - \frac{e^2}{4} - \frac{3}{64}e^4 - \cdots\right],$$

by expanding $\sqrt{1 - e^2\cos^2\phi}$ into a series and integrating term by term.

Ex. 7. Find the length of arc of the curve

$$x = e^{\theta}\sin\theta,\ y = e^{\theta}\cos\theta$$

from $\theta = 0$ to $\theta = \theta_1$.

160. Area of surface of revolution. Let AQ be a continuous arc of a curve whose equation is expressed in rectangular coördinates x and y. It is required to determine a formula for the area of the surface generated by revolving the arc AQ about the x-axis.

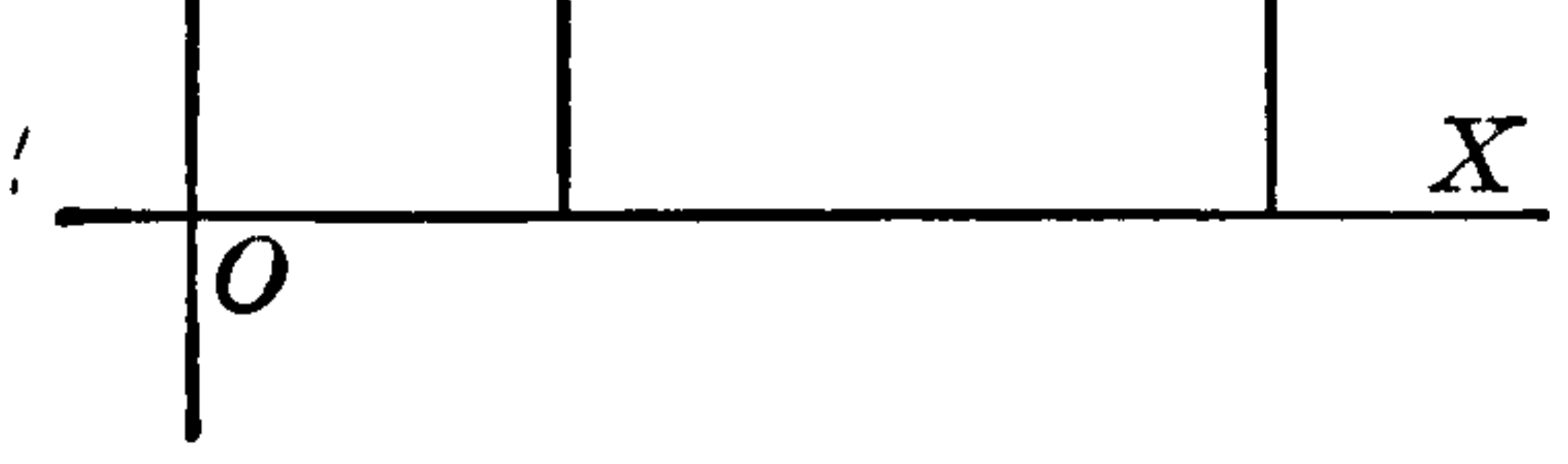

FIG. 74.

It has been shown in Art. 86, p. 140, that if S denotes the area of the surface generated by the rotation of AP (P being a variable point with coördinates (x, y)), then

$$\frac{dS}{ds} = 2\,\pi y,$$

from which $$\frac{dS}{dx} = 2\,\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

and $$\frac{dS}{dy} = 2\,\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

Hence, by integrating these two expressions,

$$\text{surface} = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_\alpha^\beta y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy, \tag{9}$$

the limiting values of x and y being the coördinates of the points A and Q.

That the result of integration is to be evaluated between the limits a and b (or α and β) is readily seen by following the suggestions made in Art. 154. For, denoting the indefinite integral

$$2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{10}$$

by $\phi(x) + C$, since the area is evidently zero when $x = a$ (*i.e.*, when the point P coincides with A) it follows that

$$\phi(a) + C = 0,$$

whence $$C = -\phi(a).$$

Moreover, when P coincides with Q the required surface is determined, and therefore

$$\text{surface generated by } AQ = \phi(b) + C = \phi(b) - \phi(a).$$

But according to Art. 148, $\phi(b) - \phi(a)$ is the definite integral obtained by evaluating (10) between the limits a and b.

In like manner it is found that the area of the surface obtained by revolving AQ about the y-axis is

$$2\pi\int_\alpha^\beta x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \tag{11}$$

In each of the above cases there is a choice of two formulas for the area. That one should be selected which can most easily be integrated.

Ex. 1. Find the surface of the catenoid obtained by revolving the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ about the y-axis, from $x = 0$ to $x = a$.

Since
$$\frac{dy}{dx} = \tfrac{1}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}}),$$
it follows that
$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{(e^{\frac{x}{a}} + e^{-\frac{x}{a}})^2}{4},$$

and hence, by using the second formula of (11), the required surface has the area
$$2\pi\int_0^a x(e^{\frac{x}{a}} + e^{-\frac{x}{a}})dx.$$

Ex. 2. Find the surface obtained by revolving about the y-axis the quarter of the circle $x^2 + y^2 + 2x + 2y + 1 = 0$ contained between the points where it touches the coördinate axes.

Ex. 3. Find the surface generated by revolving the parabola $y^2 = 4px$ about the x-axis from the origin to the point $(p, 2p)$.

Ex. 4. Find the surface generated by the revolution about the y-axis of the same arc as in Ex. 3.

Ex. 5. Find the surface generated by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

(*a*) about its major axis (the prolate spheroid);
(*b*) about its minor axis (the oblate spheroid).

Ex. 6. Find the surface generated by the revolution of the cardioid $\rho = a(1 + \cos\theta)$ about the polar axis.

Regarding the figure as referred in the first place to rectangular axes such that $x = \rho\cos\theta$, $y = \rho\sin\theta$ we have
$$\text{surface} = 2\pi\int y\,ds = 2\pi\int_0^\pi \rho\sin\theta\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}d\theta,$$
since
$$ds = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}d\theta \text{ by Art. 87.}$$

Ex. 7. Find the surface of the cone obtained by revolving that portion of the line $\frac{x}{a} + \frac{y}{b} = 1$ which is intercepted by the coördinate axes,

(α) about the x-axis;
(β) about the y-axis.

Ex. 8. Find the surface of the sphere obtained by revolving the circle $\rho = 2\,a \cos\theta$ about the polar axis. [Cf. Ex. 6.]

Ex. 9. Find the surface generated by the revolution of a complete arch of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about the x-axis.

there

161. Volume of solid of revolution. Let the plane area, bounded by an arc PQ of a given curve (referred to rectangular axes) and the ordinates at the extremities P and Q, be revolved about the x-axis. It is required to find the volume of the solid so generated.

Let the figure $APQB$ be divided into n strips of width Δx by means of the ordinates A_1P_1, A_2P_2, $\cdots$, $A_{n-1}P_{n-1}$. In revolving about the x-axis, the rectangle APR_1A_1 generates a cylinder of altitude Δx, the area of whose base is $\pi \cdot \overline{AP}^2$. Hence

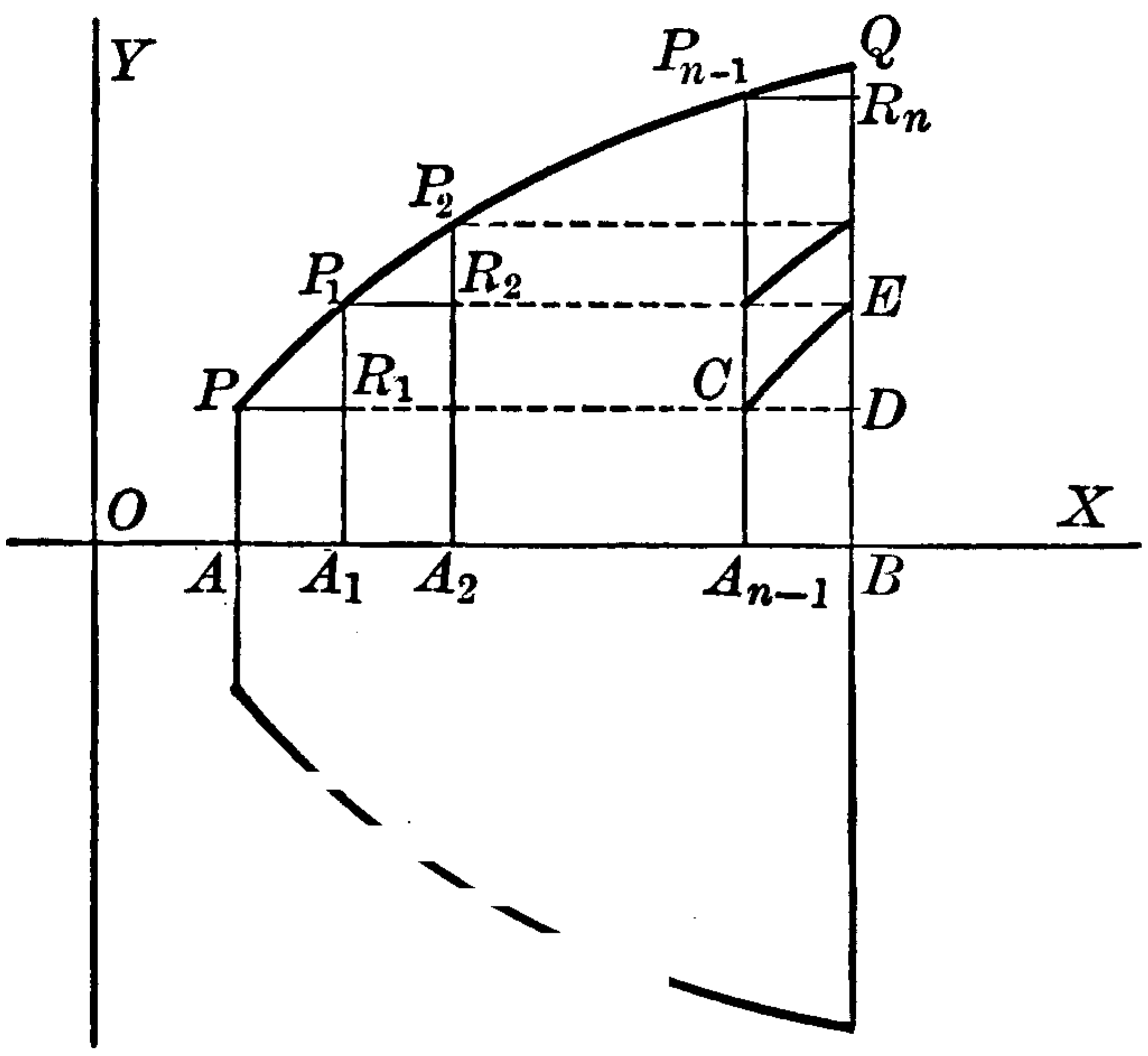

FIG. 75.

$$\text{volume of cylinder} = \pi \cdot \overline{AP}^2 \cdot \Delta x.$$

The volume of this cylinder is less than that generated by the strip APP_1A_1 by the amount contained in the ring generated by the triangular piece PR_1P_1. Imagine this ring pushed in the direction parallel to the x-axis until it occupies the position of the ring generated by CDE. If every other neglected portion (such as is generated by $P_{i-1}P_iR_i$) is treated in like manner, it is evident that their sum is less than the volume generated by the strip $A_{n-1}P_{n-1}QB$, and hence has zero for limit as Δx approaches zero. Therefore the sum of the n cylinders generated by the interior rectangles of the plane, viz.,

$$\pi(\overline{AP}^2 + \overline{A_1P_1}^2 + \cdots + \overline{A_{n-1}P_{n-1}}^2)\Delta x,$$

has for limit the volume required. But the limit of this sum is [by formula (5), p. 250] the definite integral $\int_a^b \pi y^2 dx$, and hence

$$\text{volume} = \pi \int_a^b y^2 dx.$$

The same result is readily obtained by integrating the expression for the derivative of volume. [Art. 85, p. 140.]

The volume generated by revolution about the y-axis is found by a like process to be expressed by the definite integral

$$\pi \int_\alpha^\beta x^2 dy,$$

in which α and β are the values of y at the extremities of the given arc.

Ex. 1. Find the volume of the oblate spheroid obtained by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.

Ex. 2. Find the volume of the sphere obtained by revolving the circle $x^2 + (y - k)^2 = r^2$ about the y-axis.

Ex. 3. The arc of the hyperbola $xy = k^2$, extending from the vertex to infinity is revolved about its asymptote. Find the volume generated. What is the volume generated by revolving the same arc about the other asymptote?

Ex. 4. Find the entire volume obtained by rotating the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about either axis.

Ex. 5. Find the volume obtained by the revolution of that part of the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ intercepted by the coördinate axes about one of those axes.

✓ Ex. 6. Find the volume generated by the revolution of the witch $y = \frac{8a^3}{x^2 + 4a^2}$ about the x-axis.

Ex. **7**. Find the volume generated by the revolution of the witch about the y-axis, taking the portion of the curve from the vertex $(x = 0)$ to the point (x_1, y_1).

What is the limit of this volume as the point (x_1, y_1) moves toward infinity?

Ex. **8**. Find the volume obtained by revolving a complete arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

$$\text{Volume} = \pi\int_0^{2\pi a} y^2 dx = \pi a^3\int_0^{2\pi}(1 - \cos\theta)^3 d\theta.$$

Ex. **9**. Find the volume obtained by revolving the cardioid $\rho = a(1 - \cos\theta)$ about the polar axis.

Assume $$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

Then $$dx = d(\rho\cos\theta) = d[a(1 - \cos\theta)\cos\theta]$$
$$= a\sin\theta(-1 + 2\cos\theta)d\theta.$$

Hence
$$\text{Volume} = \pi\int y^2 dx = -\pi a^3\int_0^{\pi}\sin^3\theta(1 - \cos\theta)^2(1 - 2\cos\theta)d\theta.$$

Ex. **10**. A quadrant of a circle revolves about its chord. Find the volume of the spindle so generated.

The equation of the circle being taken in the form

$$\left(x - \frac{r}{\sqrt{2}}\right)^2 + \left(y - \frac{r}{\sqrt{2}}\right)^2 = r^2,$$

the x-axis can be assumed as the axis of rotation. The ordinates of the rotated arc are determined by the formula

$$y = \frac{r}{\sqrt{2}} - \sqrt{r^2 - \left(x - \frac{r}{\sqrt{2}}\right)^2}.$$

162. Miscellaneous applications. In the preceding article the volume of the solid of revolution is shown to be the limit of the sum of the volumes of a series of cylindrical plates of thickness Δx. The notion here involved is, with suitable modifications, applicable to a variety of problems. The following examples (excepting Exs. 6, 9, 10) are illustrations of this principle.

Ex. 1. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Imagine the solid divided into a number of thin plates by means of planes perpendicular to the x-axis and at equal distances Δx from each other. Regard the volume of each plate as approximately that of an elliptic cylinder of altitude Δx. The base of the cylinder will be the ellipse in which the ellipsoid is intersected by one of the cutting planes. If the equation of this plane be denoted by $x=\lambda$, the equation of the elliptic base of the cylinder is (in y, z coördinates)

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{\lambda^2}{a^2},$$

or

$$\frac{y^2}{\frac{b^2}{a^2}(a^2-\lambda^2)}+\frac{z^2}{\frac{c^2}{a^2}(a^2-\lambda^2)}=1.$$

The semi-axes of the ellipse are

$$\frac{b}{a}\sqrt{a^2-\lambda^2} \quad\text{and}\quad \frac{c}{a}\sqrt{a^2-\lambda^2}.$$

Since the area of the ellipse is the product of the semi-axes multiplied by π (Ex. 13, p. 263), it follows that

$$\text{area of elliptic base} = \pi\cdot\frac{b}{a}\sqrt{a^2-\lambda^2}\cdot\frac{c}{a}\sqrt{a^2-\lambda^2}$$

$$=\frac{\pi bc}{a^2}(a^2-\lambda^2),$$

and

$$\text{volume of elliptic cylinder} = \frac{\pi bc}{a^2}(a^2-\lambda^2)\,\Delta\lambda,$$

($\Delta\lambda$ being used in place of Δx since $x=\lambda$).

The result of summing all such terms and taking the limit as $\Delta\lambda$ approaches zero is equivalent to the definite integral

$$\int_{-a}^{a}\frac{\pi bc}{a^2}(a^2-\lambda^2)d\lambda = 2\int_0^a\frac{\pi bc}{a^2}(a^2-\lambda^2)\,d\lambda.$$

On account of the ellipsoid being symmetrical with respect to the plane $x=0$, the limits 0 and a include one half the required volume and hence instead of using limits $-a$ and $+a$ it is more convenient to write the definite integral in the above form.

Ex. 2. Find by the method of Ex. 1 the volume of the elliptic cone

$$\frac{y^2}{a^2}+\frac{z^2}{b^2}=(x-1)^2,$$

measured from the yz-plane as base to the vertex (1, 0, 0).

Ex. 3. Find the volume of a pyramid of altitude h and of base-area A.

Ex. 4. Given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. On the major axis a plane rectangle $ABCD$ is constructed perpendicular to the plane of the ellipse. Through any point P of the line CD a plane is constructed perpendicular to CD. The two points R and S in which the latter plane meets the ellipse are joined to P by straight lines. The totality of all lines so determined form a ruled surface called a conoid. Given $AC=p$, find the volume of the above conoid.

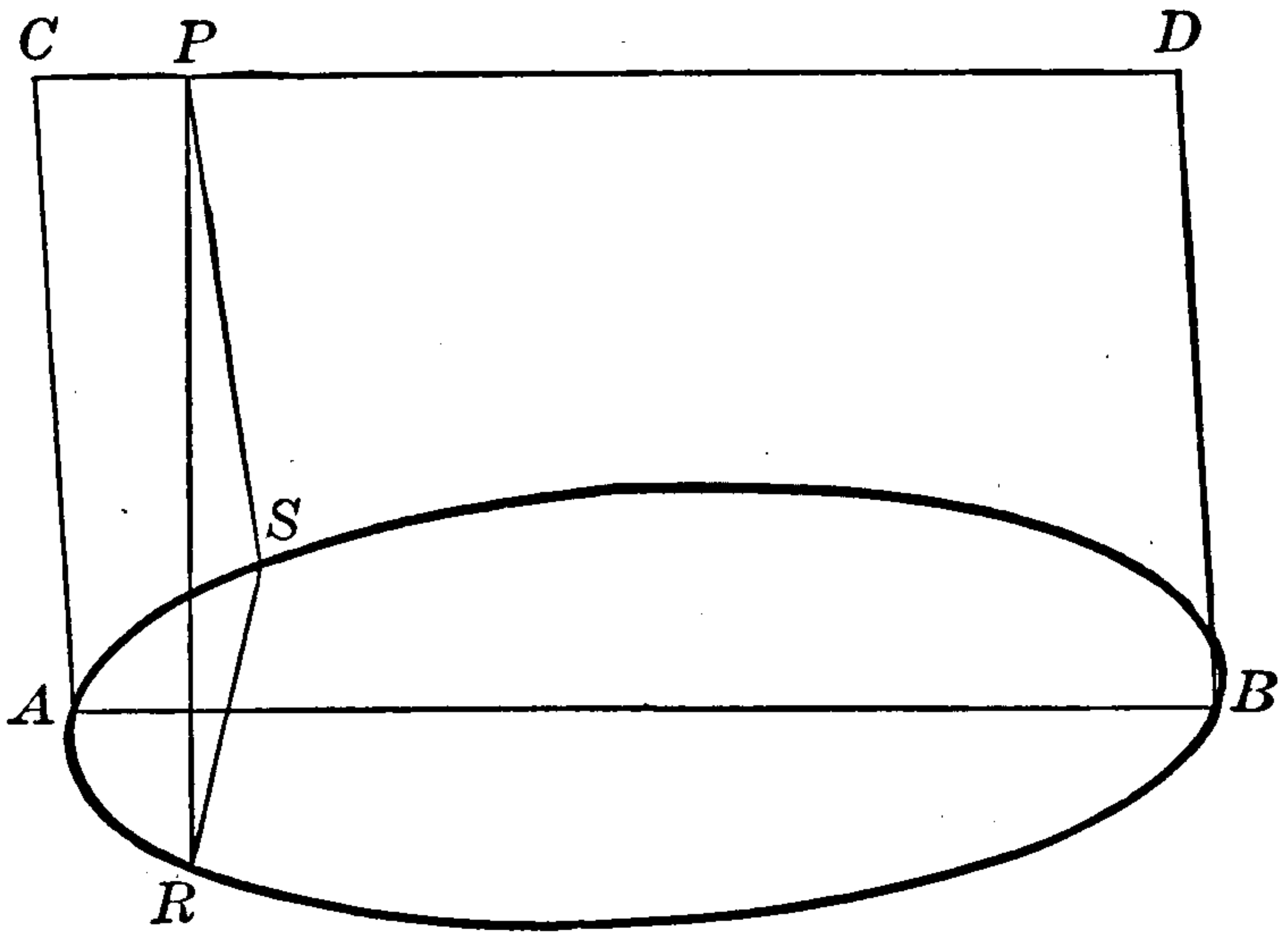

Fig. 76.

Ex. 5. A rectangle moves from a fixed point P parallel to itself, one side varying as the distance from P, and the other as the square of this distance. At the distance of 2 ft., the rectangle becomes a square of 3 ft. on each side. What is the volume generated?

Ex. 6. A string AB of length a has a weight attached at B. The other extremity A moves along a straight line OX drawing the weight in a rough horizontal plane XOY. The path traced by the point B is called the *tractrix*. What is its equation?

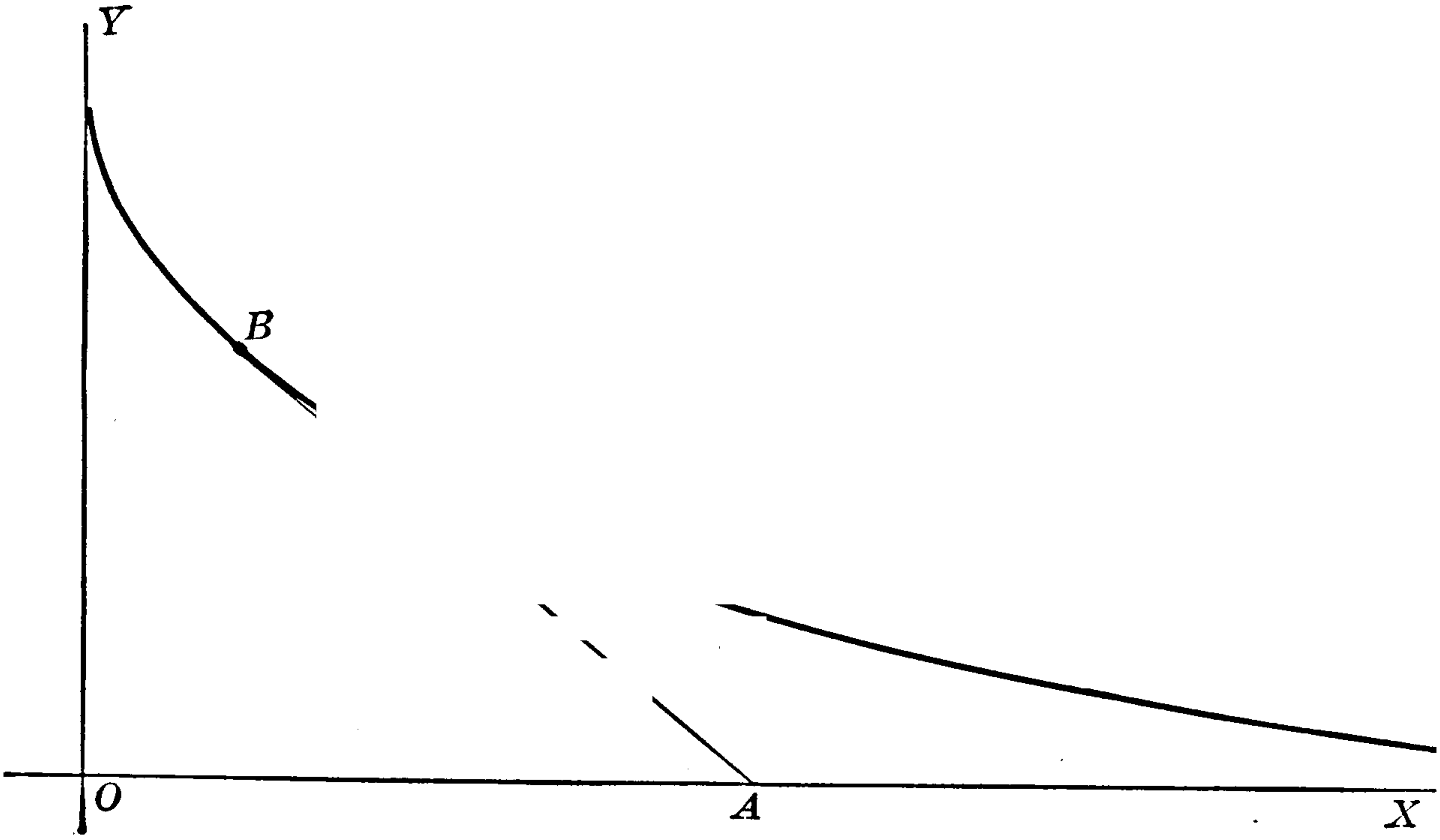

Fig. 77.

Let OY be the initial position of the string and AB any intermediate position. Since at every instant the force is exerted on the weight B in the direction of the string BA, the motion of the point must be in the same direction; that is, the direction of the tractrix at B is the same as that of the line BA and hence BA is a tangent to the curve. The expression for the tangent length is (Art. 79, p. 130)

$$\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} = y\sqrt{\left(\frac{dx}{dy}\right)^2+1} = a.$$

Solving for $\frac{dx}{dy}$,

$$\frac{dx}{dy} = \sqrt{\frac{a^2-y^2}{y^2}}.$$

Integrating with respect to y,

$$x = \int \frac{\sqrt{a^2-y^2}}{y}\,dy = \sqrt{a^2-y^2} - a\log\frac{a+\sqrt{a^2-y^2}}{y} + C.$$

The constant of integration is determined by the assumption that $(0, a)$ is the starting point of the curve. Substituting these coördinates in the above equation we find $C = 0$.

✓

Ex. 7. A woodman fells a tree 2 feet in diameter, cutting halfway through on each side. The lower face of each cut is horizontal, and the upper face makes an angle of 60° with the lower. How much wood does he cut out?

Ex. 8. The center of a square moves along a diameter of a given circle of radius a, the plane of the square being perpendicular to that of the circle, and its magnitude varying in such a way that two opposite vertices move on the circumference of the circle. Find the volume of the solid generated.

Ex. 9. The *equiangular spiral* is a curve so constructed that the angle between the radius vector to any point and the tangent at the same point is constant. Find its equation.

Ex. 10. Determine the curve having the property that the line drawn from the foot of any ordinate of the curve perpendicular to the corresponding tangent is of constant length a.

If the angle which the tangent makes with the x-axis be denoted by ϕ, it is at once evident that

$$\frac{a}{y} = \cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

From this follows

$$\frac{x}{a} = \log(y+\sqrt{y^2-a^2}) + C.$$

When the tangent is parallel to the x-axis the ordinate itself is the perpendicular a. If this ordinate be chosen for the y-axis the point $(0, a)$ is a point of the curve, and hence

$$C = -\log a.$$

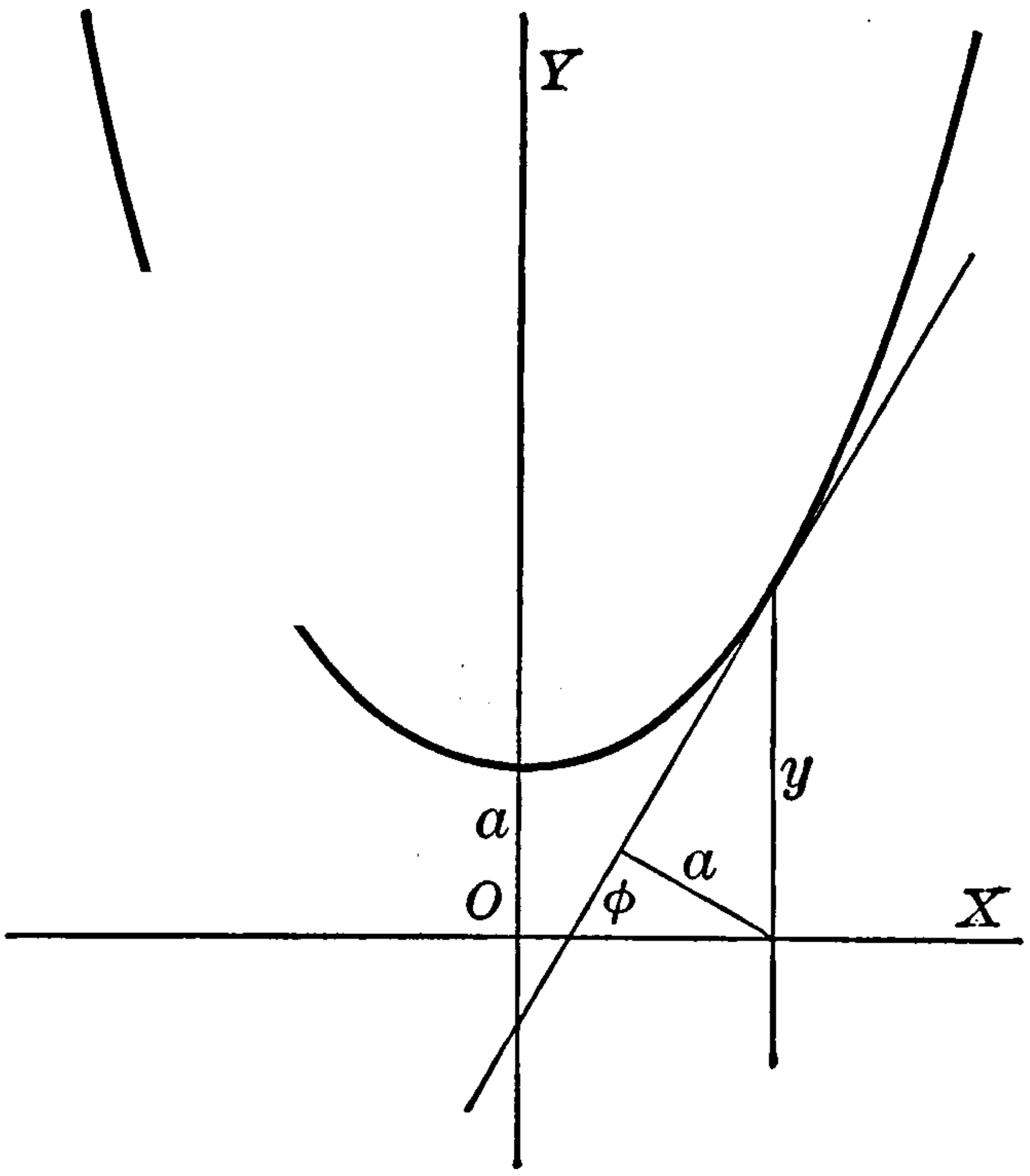

FIG. 78.

The equation can accordingly be written

$$(1) \qquad \frac{y+\sqrt{y^2-a^2}}{a} = e^{\frac{x}{a}}.$$

From this follows, by taking the reciprocal of both members,

$$\frac{a}{y+\sqrt{y^2-a^2}} = e^{-\frac{x}{a}},$$

or, rationalizing the denominator,

$$(2) \qquad \frac{y-\sqrt{y^2-a^2}}{a} = e^{-\frac{x}{a}}.$$

Adding (1) and (2) and dividing by $\frac{2}{a}$,

$$y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}),$$

which is the equation of the catenary.

Ex. **11.** A right circular cone having the angle 2θ at the vertex has its vertex on the surface of a sphere of radius a and its axis passing through the center of the sphere. Find the volume of the portion of the sphere which is exterior to the cone.

Ex. **12.** Find the volume of the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$ cut off by the plane $z = c$.

EXERCISES ON CHAPTER VII

✓ **1.** Find the area bounded by the hyperbola $xy = a^2$, the x-axis, and the two ordinates $x = a$, $x = na$.

From the result obtained, prove that the area contained between an infinite branch of the curve and its asymptote is infinite.

2. Find the area contained between the curves $y^3 = x$ and $x^3 = y$.

3. Find the area of the evolute of the ellipse

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

4. Find the area bounded by the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coördinate axes.

5. Find the area contained between the curve

$$y^2 = x^2\frac{a - x}{a + x}$$

and its asymptote $x = -a$.

[HINT. The integration may be facilitated by the substitution $x = a\cos\theta$.]

6. Find the area between the curve $y^2(y^2 - 2) = x - 1$ and the coördinate axes.

✓ **7.** Find the area common to the two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

8. If (a, α) and (b, β) are two pairs of values of x and y, the formula for integration by parts gives

$$\int_a^b y\,dx = b\beta - a\alpha - \int_\alpha^\beta x\,dy.$$

Interpret this result geometrically in terms of area.

9. Find the area bounded by the logarithmic (or equiangular) spiral $\rho = e^{a\theta}$ and the two radii ρ_1, ρ_2.

10. Find the length of an arc of the spiral of Archimedes $\rho = a\theta$ between the points (ρ_1, θ_1), (ρ_2, θ_2).

✓ **11.** Find the surface of the ring generated by revolving the circle $x^2 + (y - k)^2 = a^2$ $(k > a)$ about the x-axis.

12. Find the volume of the ring defined in Ex. 11.

✓ **13.** Find the volume obtained by revolving about the x-axis that portion of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ limited by the points $(-x_1, y_1)$ and (x_1, y_1).

E **14.** Find the entire volume generated by the revolution of the cissoid

$$y^2 = \frac{x^3}{2a - x}$$

about its asymptote.

2a [illegible] π

[Hint. For the purpose of integration, assume

$$x = 2a\sin^2\theta, \text{ whence } y = \frac{2a\sin^3\theta}{\cos\theta}.]$$

E **15.** Find the surface generated by the rotation of the involute of the circle $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$ about the x-axis from $t = 0$ to $t = t_1$.

16. Find the volume generated by the revolution of the tractrix (see Ex. 6, p. 281) about the positive x-axis.

17. Find the area of the surface of revolution described in Ex. 16.

✓ **18.** Find the length of the tractrix from the cusp (the point $(0, a)$) to the point (x_1, y_1).

CHAPTER VIII

SUCCESSIVE INTEGRATION

163. Functions of a single variable. Thus far we have considered the problem of finding the function y of x when $\frac{dy}{dx}$ only is given. It is now proposed to find y when its nth derivative $\frac{d^ny}{dx^n}$ is given.

The mode of procedure is evident. First find the function $\frac{d^{n-1}y}{dx^{n-1}}$ which has $\frac{d^ny}{dx^n}$ for its derivative. Then, by integrating the result, determine $\frac{d^{n-2}y}{dx^{n-2}}$, and so on until after n successive integrations the required result is found. As an arbitrary constant should be added after each integration in order to obtain the most general solution, the function y will contain n arbitrary constants.

Ex. 1. Given $\frac{d^3y}{dx^3}=\frac{1}{x^3}$, find y.

Integration of $\frac{1}{x^3}$ with respect to x gives

$$\frac{d^2y}{dx^2}=-\frac{1}{2x^2}+C_1.$$

Integrating a second time,

$$\frac{dy}{dx}=\frac{1}{2x}+C_1x+C_2;$$

and finally $$y=\tfrac{1}{2}\log x+\tfrac{1}{2}C_1x^2+C_2x+C_3.$$

The triple integration required in this example will be symbolized by

$$\int\int\int\frac{1}{x^3}(dx)^3,$$

which will be called the *triple integral* of $\frac{1}{x^3}$ with respect to x.

Ex. **2.** Determine the curves having the property that the radius of curvature at any point P is proportional to the cube of the secant of the angle which the tangent at P makes with a fixed line.

If a system of rectangular axes be chosen with the given line for x-axis, it follows from equation (6), p. 164, and from Art. 10, that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = a\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}},$$

in which a is an arbitrary constant. This equation reduces to

$$\frac{d^2y}{dx^2} = a,$$

from which follows

$$y = \int\int a(dx)^2 = a\left[\frac{x^2}{2} + C_1x + C_2\right],$$

C_1 and C_2 being constants of integration. Hence the required curves are the parabolas having axes parallel to the y-axis.

The existence of the two arbitrary constants C_1, C_2 in the preceding equation makes it possible to impose further conditions. Suppose, for example, it be required to determine the curve having the property already specified, and having besides a maximum (or a minimum) point at (1, 0).

Since at such a point $\frac{dy}{dx} = 0$, it follows that

$$0 = a(1 + C_1),$$

whence $$C_1 = -1.$$

Also, by substituting (1, 0) in the equation of the curve,

$$0 = a(\tfrac{1}{2} - 1 + C_2),$$

from which $$C_2 = \tfrac{1}{2}.$$

Accordingly the required curve is

$$y = \frac{a}{2}(x-1)^2.$$

Ex. **3.** Find the equation (in rectangular coördinates) of the curves having the property that the radius of curvature is equal to the cube of the tangent length.

[HINT. Take y as the independent variable.]

Ex. 4. A particle moves along a path in a plane such that the slope of the line tangent at the moving point changes at a rate proportional to the reciprocal of the abscissa of that point. Find the equation of the curve.

Ex. 5. A particle starting at rest from a point P moves under the action of a force such that the acceleration (cf. Ex. 14, p. 111) at each instant of time is proportional to (is k times) the square root of the time. How far will the particle move in the time t?

164. Integration of functions of several variables. When functions of two or more variables are under consideration, the process of differentiation can in general be performed with respect to any one of the variables, while the others are treated as constant during the differentiation. A repetition of this process gives rise to the notion of successive partial differentiation with respect to one or several of the variables involved in the given function. [Cf. Arts. 68, 72.]

The reverse process readily suggests itself, and presents the problem: *Given a partial (first, or higher) derivative of a function of several variables with respect to one or more of these variables, to find the original function.*

This problem is solved by means of the ordinary processes of integration, but the added constant of integration has a new meaning. This can be made clear by an example.

Suppose u is an unknown function of x and y such that

$$\frac{\partial u}{\partial x} = 2x + 2y.$$

Integrate this with respect to x alone, treating y at the same time as though it were constant. This gives

$$u = x^2 + 2xy + \phi,$$

in which ϕ is an added constant of integration. But since y is regarded as constant during this integration there is nothing to prevent ϕ from depending on it. This dependence may be indicated by writing $\phi(y)$ in the place of ϕ. Hence the most general function having $2x+2y$ for its partial derivative with respect to x is

$$u = x^2 + 2xy + \phi(y),$$

in which $\phi(y)$ is *an entirely arbitrary function of* y.

Again, suppose

$$\frac{\partial^2 u}{\partial x \partial y} = x^2 y^2.$$

Integrating first with respect to y, x being treated as though it were constant during this integration, we find

$$\frac{\partial u}{\partial x} = \frac{1}{3} x^2 y^3 + \psi(x),$$

where $\psi(x)$ is an arbitrary function of x, and is to be regarded as an added constant for the integration with respect to y.

Integrate the result with respect to x, treating y as constant. Then

$$u = \tfrac{1}{9} x^3 y^3 + \Psi(x) + \Phi(y).$$

Here $\Phi(y)$, the constant of integration with respect to x, is an arbitrary function of y, while

$$\Psi(x) = \int \psi(x) dx.$$

Since $\psi(x)$ is an arbitrary function of x, so also is $\Psi(x)$.

165. Integration of a total differential. The total differential of a function u depending on two more variables has been defined (Art. 69) by the formula

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

The question now presents itself: *Given a differential expression of the form*

$$P\,dx + Q\,dy, \qquad (1)$$

wherein P and Q are functions of x and y, does there exist a function u of the same variables having (1) *for its total differential?*

It is easy to see that in general such a function does not exist. For, in order that (1) may be a total differential of a function u, it is evidently necessary that P and Q have the form

$$P=\frac{\partial u}{\partial x},\ Q=\frac{\partial u}{\partial y}. \tag{2}$$

What relation, then, must exist between P and Q in order that the conditions (2) may be satisfied? This is easily found as follows: Differentiate the first equation of (2) with respect to y, and the second with respect to x. This gives

$$\frac{\partial P}{\partial y}=\frac{\partial^2 u}{\partial y\partial x},\ \frac{\partial Q}{\partial x}=\frac{\partial^2 u}{\partial x\partial y},$$

from which follows

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}. \tag{3}$$

This is the relation sought.

The next step is to find the function u by integration. It is easier to make this process clear by an illustration.

Given $(2x+2y+2)\,dx+(2y+2x+2)\,dy$,

find the function u having this as its total differential.

Since $P=2x+2y+2,\ Q=2y+2x+2$,

it is found by differentiation that

$$\frac{\partial P}{\partial y}=2 \text{ and } \frac{\partial Q}{\partial x}=2,$$

and hence the necessary relation (3) is satisfied.

From (2) it follows that

$$\frac{\partial u}{\partial x}=2x+2y+2.$$

Integrating this with respect to x alone,

$$u = x^2 + 2xy + 2x + \phi(y). \tag{4}$$

It now remains to determine the function $\phi(y)$ so that

$$\frac{\partial u}{\partial y}(= Q) = 2y + 2x + 2. \tag{5}$$

Differentiate (4) with respect to y alone, whence

$$\frac{\partial u}{\partial y} = 2x + \phi'(y),$$

where $\phi'(y)$ denotes the derivative of $\phi(y)$ with respect to y. The comparison of this result with (5) gives

$$2y + 2x + 2 = 2x + \phi'(y),$$

or

$$\phi'(y) = 2y + 2, \tag{6}$$

whence, by integrating with respect to y,

$$\phi(y) = y^2 + 2y + C,$$

in which C is an arbitrary constant with respect to both x and y.

Hence $$u = x^2 + 2xy + 2x + y^2 + 2y + C.$$

It is to be remarked that in integrating (6) we integrate exactly those terms in Q which do not contain x. Hence the following rule may be formulated for integrating a total differential:

Integrate P with respect to x alone, treating y as constant. Then integrate with respect to y those terms of Q which contain y but do not contain x, and add the result, together with an arbitrary constant C, to the terms already obtained.

It is evident that it would be equally well to first integrate Q with respect to y, and then integrate those terms of P which contain x alone with respect to x, and add the two results.

EXERCISES

Determine in each of the following cases the function u having the given expression for its total differential:

1. $y\,dx + x\,dy$.

2. $\sin x \cos y\,dx + \cos x \sin y\,dy$.

3. $y\,dx - x\,dy$.

4. $\dfrac{y\,dx - x\,dy}{xy}$.

5. $(3x^2 - 3ay)\,dx + (3y^2 - 3ax)\,dy$.

6. $\dfrac{y\,dx}{x^2 + y^2} - \dfrac{x\,dy}{y^2 + x^2}$.

7. $(2x^2 + 2xy + 5)\,dx + (x^2 + y^2 - y)\,dy$.

8. $(x^4 + y^4 + x^2 - y^2)\,dx + (4y^3x - 2xy + y - y^2 + 2)\,dy$.

166. Multiple integrals. The integration of $\dfrac{\partial^2 u}{\partial x\,\partial y}$ was considered in Art. 164. If $F(x, y)$ be written for the given function, the required integration will be represented by the symbol

$$u = \int\!\!\int F(x, y)\,dx\,dy,$$

and the function sought will be called the ***double integral*** of $F(x, y)$ with respect to x and y.

Likewise $$\int\!\!\int\!\!\int F(x, y, z)\,dx\,dy\,dz$$

will be called the ***triple integral*** of $F(x, y, z)$. It represents the function u whose third partial derivative $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$ is the given function $F(x, y, z)$. It will be understood in what follows that the order of integration is from right to left, that is, we integrate first with respect to the right hand variable z, then with respect to y, and lastly with respect to x.

Such integrals (double, triple, etc.) will be referred to in general as multiple integrals.

167. Definite multiple integrals. The idea of a multiple integral may be further extended so as to include the notion of a *definite multiple integral* in which limits of integration may be assigned to each variable.

Thus the integral $\int_a^b\int_0^2 x^2y^3\,dx\,dy$ will mean that x^2y^3 is to be integrated first with respect to y between the limits 0 and 2. This gives

$$\int_0^2 x^2y^3\,dy = 4\,x^2.$$

The result so obtained is to be integrated with respect to x between the limits a and b, which leads to

$$\int_a 4\,x^2\,dx = \tfrac{4}{3}(b^3 - a^3)$$

as the value of the given definite double integral.

In general the expression

$$\int_a^{a'}\int_b^{b'} F(x, y)\,dx\,dy$$

will be used as the symbol of a definite double integral. It will be understood that the integral signs with their attached limits are always to be read from right to left, so that in the above integral the limits for y are b and b', while those for x are a and a'.

Since x is treated as constant in the integration with respect to y, the limits for y may be functions of x. Consider, for example, the integral $\int_0^1\int_{\sqrt{x}}^x xy\,dx\,dy$. The first integration (with respect to y) gives

$$\int_{\sqrt{x}}^x xy\,dy = x\left[\frac{y^2}{2}\right]_{\sqrt{x}}^x = x\left(\frac{x^2}{2} - \frac{x}{2}\right) = \frac{x^3 - x^2}{2}.$$

By integrating this result with respect to x between limits 0 and 1 the given integral is found to have the value $-\frac{1}{24}$.

EXERCISES

Evaluate the following integrals:

1. $\int_1^e \int_0^{\frac{\pi}{4x}} \sec^2(xy)\, dx\, dy.$

2. $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \sin\theta\, d\theta\, dr.$

3. $\int_0^b \int_y^{10y} \sqrt{xy - y^2}\, dy\, dx.$

4. $\int_1^2 \int_0^z \int_0^{x\sqrt{3}} \frac{x\, dz\, dx\, dy}{x^2 + y^2}.$

✓

168. Plane areas by double integration. The area bounded by a plane curve (or by several curves) can be readily expressed in the form of a definite double integral. An illustrative example will explain the method.

Ex. 1. Find by double integration the area of the circle $(x - a)^2 + (y - b)^2 = r^2$.

Imagine the given area divided into rectangles by a series of lines parallel to the y-axis at equal distances Δx, and a series of lines parallel to the x-axis at equal distances Δy.

The area of one of these rectangles is $\Delta y \cdot \Delta x$. This is called the element of area. The sum of all the rectangles interior to the circle will be less than the area required by the amount contained in the small subdivisions which border the circumference of the circle. By a method exactly analogous to that used in Art. 149, it is easy to show that the sum of these neglected portions has a zero limit when Δx and Δy are both made to approach zero.

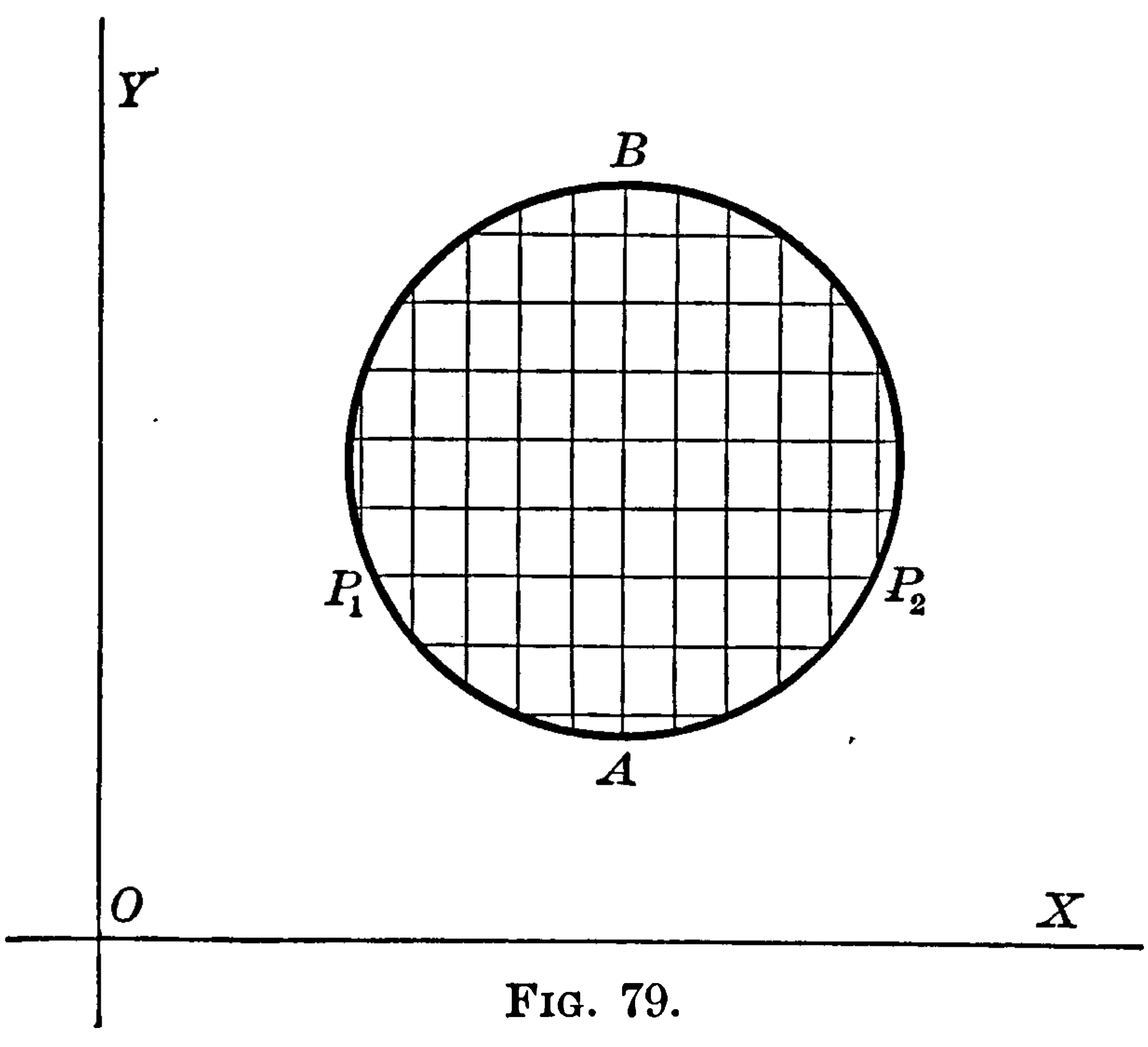

FIG. 79.

To find the value of the limit of the sum of all the rectangles within the circle it is convenient to first add together all those which are contained between two consecutive parallels. Let P_1P_2 be one of these parallels having the direction of the x-axis. Then y remains constant while x varies from $a - \sqrt{r^2 - (y - b)^2}$ (the value of the abscissa at P_1) to $a + \sqrt{r^2 - (y - b)^2}$ (the value at P_2). The limit as Δx approaches zero of the sum of rectangles in the strip from P_1P_2 is evidently

$$(1) \qquad \Delta y\,[\text{limit of sum } (\Delta x + \Delta x + \cdots)] = \Delta y \int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx.$$

Now find the limit of the sum of all such strips contained within the circle. This requires the determination of the limit of the sum of terms such as (1) for the different values of y corresponding to the different strips. Since y begins at the lowest point A with the value $b - r$, and increases to $b + r$, the value reached at B, the final expression for the area is

$$\int_{b-r}^{b+r} dy \int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx = \int_{b-r}^{b+r}\int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dy\, dx.$$

Integrating first with respect to x,

$$\int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx = x\,\Big]_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} = 2\sqrt{r^2 - (y - b)^2}.$$

This result is then integrated with respect to y, giving

$$\int_{b-r}^{b+r} 2\sqrt{r^2 - (y - b)^2}\,dy = (y - b)\sqrt{r^2 - (y - b)^2} + r^2 \sin^{-1}\frac{y - b}{r}\Big]_{b-r}^{b+r} = \pi r^2.$$

If the summation had begun by adding the rectangles in a strip parallel to the y-axis, and then adding all of these strips, the expression for the area would take the form

$$\int_{a-r}^{a+r}\int_{b-\sqrt{r^2-(x-a)^2}}^{b+\sqrt{r^2-(x-a)^2}} dx\, dy.$$

It is seen from this last result that the order of integration in a double integral can be changed if the limits of integration be properly modified at the same time.

Ex. **2.** Find the area which is included between the two parabolas

$$y^2 = 9\,x \text{ and } y^2 = 72 - 9\,x.$$

Ex. **3.** Find the area common to the two circles

$$x^2 - 8\,x + y^2 - 8\,y + 28 = 0,$$

$$x^2 - 8\,x + y^2 - 4\,y + 16 = 0.$$

169. Volumes. The volume bounded by one or more surfaces can be expressed as a triple integral when the equations of the bounding surfaces are given.

Let it be required to find the volume bounded by the surface *ABC* (Fig. 80) whose equation is $z = f(x, y)$, and by the three coördinate planes.

Imagine the figure divided into small equal rectangular parallelopipeds by means of three series of planes, the first series parallel to the yz-plane at equal distances Δx, the second parallel to the xz-plane at equal distances Δy, and the third parallel to the xy-plane at equal distances Δz. The volume of such a rectangular solid is $\Delta x \Delta y \Delta z$; it is called the element of volume. The limit of the sum of all such elements contained in *OABC* is the volume required, provided that the bounding surface *ABC* is continuous.

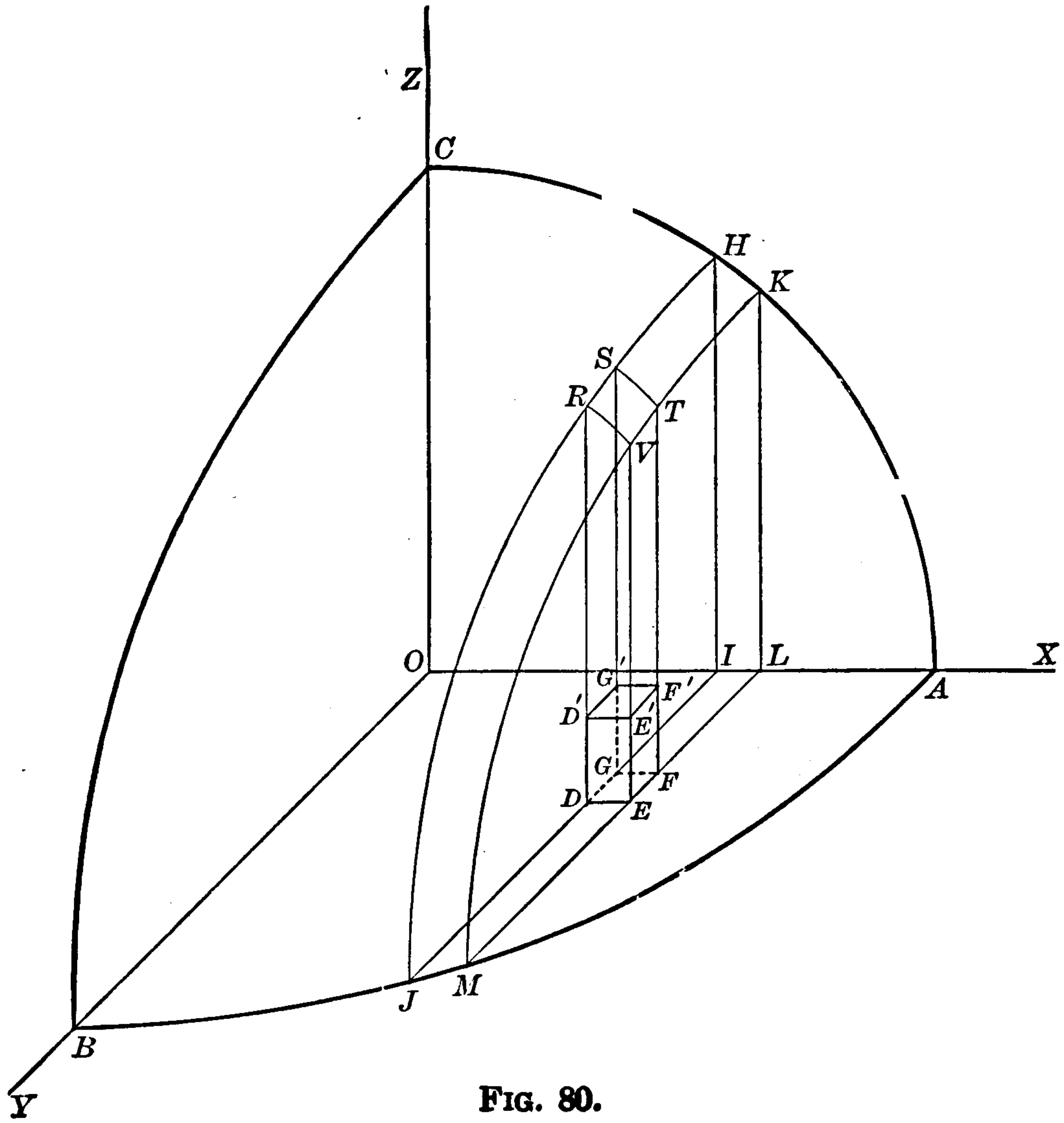

FIG. 80.

(The reader can easily show that the sum of the neglected portions is less than the volume of the largest plate formed by two consecutive parallel planes and that its limit is therefore zero.)

To effect this summation, add first all the elements in a vertical column. This corresponds to integrating with respect to z (x and y remaining constant) from zero to $f(x, y)$. Then add all such vertical columns contained between two consecutive planes parallel to the yz-plane (x remaining constant), which corresponds to an integration with respect to y from $y = 0$ to the value attained on the boundary of the curve AB. This value of y is found by solving the equation $f(x, y) = 0$. Finally, add all such plates for values of x varying from zero to the value at A. The final result is expressed by the integral

$$\int_0^a \int_0^{\phi(x)} \int_0^{f(x, y)} dx\, dy\, dz,$$

in which $\phi(x)$ is the result of solving the equation $f(x, y) = 0$ for y, and a is the x-coördinate of A.

Ex. 1. Find the volume of the sphere of radius a.

The equation of the sphere is

$$x^2 + y^2 + z^2 = a^2,$$

or

$$z = \sqrt{a^2 - x^2 - y^2}.$$

Since the coördinate planes divide the volume into eight equal portions, it is sufficient to find the volume in the first octant and multiply the result by 8.

The volume being divided into equal rectangular solids as described above, the integration with respect to z is equivalent to finding the limit of the sum of all the elements contained in any vertical column. The limits of the integration with respect to z are the values of z corresponding to the bottom and the top of such a column, namely, $z = 0$, and $z = \sqrt{a^2 - x^2 - y^2}$, since the point at the top is a point on the surface of the sphere.

The limits of integration with respect to y are found to be $y = 0$ (the value at the x-axis), and $y = \sqrt{a^2 - x^2}$ (the value of y at the circumference of the circle $a^2 - x^2 - y^2 = 0$, in which the sphere is cut by the xy-plane).

Finally, the limiting values for x are zero and a, the latter being the distance from the origin to the point in which the sphere intersects the x-axis. Hence

$$V\ (=\text{volume of sphere}) = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz.$$

Integrating first with respect to z,

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dx\,dy;$$

then with respect to y,

$$V = 8\int_0^a dx\left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\right]_0^{\sqrt{a^2-x^2}}$$

$$= 8\int_0^a \frac{\pi}{4}(a^2 - x^2)dx = \frac{4\pi a^3}{3}.$$

Ex. **2**. Find the volume of one of the wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = mx$.

✓ Ex. **3**. Find the volume common to two right circular cylinders of the same radius a whose axes intersect at right angles.

Ex. **4**. Find the volume of the cylinder $(x-1)^2 + (y-1)^2 = 1$ limited by the plane $z = 0$, and the hyperbolic paraboloid $xy = z$.

Ex. **5**. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Ex. **6**. Find the volume of that portion of the elliptic paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

which is cut off by the plane $z = 0$.

ANSWERS

Page 20. Art. 9

1. $2x-2$. 2. $6x-4$. 3. $-\frac{1}{4x^2}$. 4. $4x^3-\frac{6}{x^3}$.

Page 25. Art. 11

1. $15y^2-2$. 2. $14t-4-33t^2$. 3. $12u^2-2$. 4. $4x-5$.

Page 28. Art. 13

2. Inc. from $-\infty$ to $\frac{1}{3}$; dec. from $\frac{1}{3}$ to 1; inc. from 1 to $+\infty$; $\frac{1}{3}$ and 1.
3. Two. $+1$ at $x=\frac{1}{2}\pm\sqrt{\frac{5}{12}}$; -1 at $x=\frac{1}{2}\pm\sqrt{\frac{1}{12}}$. 4. $\pm\tan^{-1}\frac{4}{97}$.

Page 29. Art. 14

2. $(6u-4)6x^2$. 3. $-\frac{1}{u^2}(10x-2)$. 4. $\left(6u-\frac{2}{3u^3}\right)\left(x^2-\frac{9}{x^4}\right)$.

Page 37. Art. 19

1. $10x^9$.
2. $-8x^{-9}$.
3. $\frac{c}{2\sqrt{x}}$.
4. $-\frac{1}{\sqrt{x^3}}-\frac{1}{9\sqrt[3]{x^2}}$.
5. $-\frac{5}{4}\sqrt[4]{x^{-9}}$.
. $n(x+a)^{n-1}$.
7. nx^{n-1}.
8. $\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$.
9. $\frac{2-6x-x^2}{(x^2+2)^2}$.
10. $\frac{3x+5}{2\sqrt{x+2}}$.
11. $\frac{\sqrt{a}(\sqrt{x}-\sqrt{a})}{2\sqrt{x}(\sqrt{x+a})(\sqrt{a}+\sqrt{x})^2}$.
12. $\frac{1}{\sqrt{1-x^2}(1-x)}$.
13. $\frac{1}{2x(1-x^2)+\sqrt{1-x^2}}$.
14. $\frac{4a^{\frac{1}{2}}+3x^{\frac{1}{2}}}{4\sqrt{x}\sqrt{a^{\frac{1}{2}}+x^{\frac{1}{2}}}}$.
15. $\frac{ny}{x\sqrt{1-x^2}}$.

16. $\dfrac{2x^3 - 4x}{(1 - x^2)^{\frac{1}{2}}(1 + x^2)^{\frac{5}{2}}}$.

17. $\dfrac{-2nx^{n-1}}{(x^n - 1)^2}$.

18. $-\dfrac{m(b + x) + n(a + x)}{(a + x)^{m+1} \cdot (b + x)^{n+1}}$.

19. $\dfrac{-2}{x^2(x^3 + 1)^{\frac{5}{3}}}$.

20. $56x^3(x^2 + 1)^{\frac{1}{3}}$.

21. $6u\dfrac{du}{dx}$.

22. $12(u^2 - u + 1)\dfrac{du}{dx}$.

23. $60u^5(1 + u^2)^2\dfrac{du}{dx}$.

24. $u + x\dfrac{du}{dx}$.

25. $(2u + 6xu)\dfrac{du}{dx} + 3u^2 + 4x^3$.

26. $\dfrac{nu^{n-1}\dfrac{du}{dx}}{(a + x)^n} - \dfrac{nu^n}{(a + x)^{n+1}}$.

27. $2ux^3w\dfrac{du}{dx} + u^2x^3\dfrac{dw}{dx} + 3u^2x^2w$.

30. $\dfrac{-b^2x}{a^2y} = -\dfrac{bx}{a\sqrt{a^2 - x^2}}$.

32. $(0, 0)$, $\left(\dfrac{4}{9a}, -\dfrac{8}{27a}\right)$, $\left(\dfrac{4}{9a}, +\dfrac{8}{27a}\right)$.

34. $\dfrac{(21u^3 - 19u)10x}{(7u^2 + 5)^{\frac{3}{2}}}$.

35. At right angles at $(3, \pm 6)$.

Page 43. Art. 24

1. $\dfrac{1}{x + a}$.

2. $\dfrac{a}{ax + b}$.

3. $\dfrac{8x - 7}{4x^2 - 7x + 2}$.

4. $\dfrac{2}{1 - x^2}$.

5. $\dfrac{4x}{1 - x^4}$.

6. $\log x + 1$.

7. $nx^{n-1}\log x + x^{n-1}$.

8. $nx^{n-1}\log x^m + mx^{n-1}$.

9. $\dfrac{x}{x^2 -- 1}$.

10. $\dfrac{1}{2(\sqrt{x} + 1)}$.

11. $\log_a e \cdot \dfrac{12x\sqrt{2 + x} - 1}{2\sqrt{2 + x}(3x^2 - \sqrt{2 + x})}$.

12. $\log_{10} e \cdot \dfrac{2x + 7}{x^2 + 7x}$.

13. $-\dfrac{y}{x \log x}$.

14. ae^{ax}.

15. $4e^{4x+5}$

16. $\dfrac{-e^{\frac{1}{1+x}}}{(1 + x)^2}$.

17. $\dfrac{e^x}{(1 + e^x)^2}$.

18. $y - 3x^2e^x$.

19. $1 - y^2$.

20. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

21. $\dfrac{1 + e_x}{x + e^x}$.

22. $nx^{n-1}a^x + x^na^x \log a$.

23. $\dfrac{\sqrt{a}}{\sqrt{x}(a - x)}$.

24. $-\dfrac{1}{x(\log x)^2}$.

25. $\dfrac{2\log x}{x}$.

26. $\dfrac{1}{x\log x}$.

27. $x^x(\log x + 1)$.

28. $a^{\log x}\dfrac{\log a}{x}$.

29. $\dfrac{-(x-1)^{\frac{3}{2}}(7x^2+30x-97)}{12(x-2)^{\frac{7}{4}}(x-3)^{\frac{10}{3}}}$.

30. $\dfrac{2+x-5x^2}{2\sqrt{1-x}}$.

31. $\dfrac{1+3x^2-2x^4}{(1-x^2)^{\frac{3}{2}}}$.

32. $5x^4(a+3x)^2(a-2x)$
$(a^2+2ax-12x^2)$.

33. $\dfrac{(x-2a)\sqrt{x+a}}{(x-a)^{\frac{3}{2}}}$.

Page 47. Art. 31

1. $7\cos 7x$.

2. $-5\sin 5x$.

3. $2x\cos x^2$.

4. $2\cos 2x\cos x - \sin 2x\sin x$.

5. $3\sin^2 x\cos x$.

6. $10x\cos 5x^2$.

7. $14\sin 7x\cos 7x$.

8. $\sec^2 x(\tan^2 x - 1)$.

9. $3\sin^2 x\cos^2 x - \sin^4 x$.

10. $\sec x(\tan x + \sec x)$.

11. $-6x(1-2x^2)\sin(1-2x^2)^2$
$\cos(1-2x^2)^2$.

12. $-20x(3-5x^2)\sec^2(3-5x^2)^2$.

13. $2\tan x\sec^2 x - 2\tan x$.

14. $\sec x$.

15. $\dfrac{\cot\sqrt{x}}{2\sqrt{x}}$.

16. $-\dfrac{1}{x^2}\log a\cdot a^{\frac{1}{x}}\cdot\sec^2(a^{\frac{1}{x}})$.

17. $n\sin^{n-1}x\sin(n+1)x$.

18. $\cos 2u\dfrac{du}{dx}$.

19. $\dfrac{mn\sin^{m-1}nx\cdot\cos(m-n)x}{\cos^{n+1}mx}$.

20. $\dfrac{2}{1+\tan x}$.

21. $\csc\theta\dfrac{d\theta}{dx}$.

22. $\cos(\sin u)\cos u\dfrac{du}{dx}$.

23. $2ae^{ax}\sin e^{ax}\cdot\cos e^{ax}$.

24. $e^x\cdot\cos e^x\cdot\log x + \dfrac{\sin e^x}{x}$.

25. $\dfrac{x\cos x^2}{\sqrt{\sin x^2}}$.

26. $y\left(\dfrac{\sin x}{x} + \cos x\log x\right)$.

27. $-8\csc^2 4x\cot 4x$.

28. $8(4x-3)\sec(4x-3)^2$
$\tan(4x-3)^2$.

29. $3x^2\sin x^3$.

30. $-2x\csc^2 x^2 + \dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$.

31. $\dfrac{y\cos xy}{1-x\cos xy}$.

32. $-\csc^2(x+y)$.

Page 49. Art. 33

1. $\dfrac{4x}{\sqrt{1-4x^4}}$.

2. $\dfrac{1}{\sqrt{1-x^2}}$.

3. $\dfrac{3}{\sqrt{6x-9x^2}}$.

4. $\dfrac{3}{\sqrt{1-x^2}}$.

5. $\dfrac{-2}{1+x^2}$.

6. $\dfrac{1}{2\sqrt{\sin^{-1}x}\,\sqrt{1-x^2}}$.

7. $\dfrac{1}{e^x+e^{-x}}$.

8. $\dfrac{-1}{x\sqrt{1-(\log x)^2}}$.

9. $\dfrac{\sec^2 x}{\sqrt{1-\tan^2 x}}$.

10. $\dfrac{1}{\sqrt{1-x^2}}$.

11. $\dfrac{2}{\sqrt{1-x^2}}$.

12. $\dfrac{-1}{x\sqrt{x^2-1}}$.

13. $\dfrac{2}{1+x^2}$.

14. $\frac{1}{2}\sqrt{1+\csc x}$.

15. $\frac{1}{2}$.

16. $\sec^2 x\cdot\tan^{-1}x+\dfrac{\tan x}{1+x^2}$.

17. $\sin^{-1}x+\dfrac{x}{\sqrt{1-x^2}}$.

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$.

19. $\dfrac{2}{\sqrt{1-x^2}}$.

20. $\dfrac{-2}{x^2+1}$.

21. $\dfrac{1}{2\sqrt{x}(x+1)}$.

22. $\dfrac{-2}{e^x+e^{-x}}$.

23. $\dfrac{n}{\cos^2 x+n^2\sin^2 x}$.

24. 2.

25. $\dfrac{2\sin x}{\sqrt{1-4\cos^2 x}}$.

26. $\dfrac{-1}{2(1+x^2)}$.

27. $\dfrac{-1}{\sqrt{1-x^2}}$.

28. 0.

Page 51. Exercises on Chapter II

1. $6x+15x^2$.

2. $\dfrac{-6}{x^3}-\dfrac{15}{x^4}$.

3. $\dfrac{3x-1}{2\sqrt{x-3}}$.

4. $\dfrac{a^2-2x^2}{\sqrt{a^2-x^2}}$.

5. $\log\sin x+x\cot x$.

6. $\dfrac{-a^3}{x^2\sqrt{a^2-x^2}}$.

7. $\dfrac{c}{x}e^{\frac{x}{c}}\left(1-\dfrac{c}{x}\right)$.

8 $\dfrac{2x^2-2x+1}{2(x-x^2)^2}$.

9. $\dfrac{e^{\sqrt{u}}}{2\sqrt{u}}\cdot\cot x$.

10. $\dfrac{1}{x}-\log a$.

11. $\dfrac{-(3x+x^3)}{(1+x^2)^{\frac{3}{2}}}$.

12. $e^x(\cos x-\sin x)$

13. $\dfrac{-1}{x\sqrt{2x-1}}$.

14. $\dfrac{4}{5+3\cos x}$.

15. $\tan^{-1}\sqrt{\dfrac{x}{a}}$.

16. $\dfrac{1}{2(1+x^2)}$.

17. $4 \tan^5 x$.

18. $\dfrac{\log x}{(1-x)^2}$.

19. $\dfrac{4}{5 + 3\cos x}$.

20. $\dfrac{x^2}{1-x^4}$.

21. $\dfrac{1}{x}\sqrt{\dfrac{x+a}{x-a}}$.

22. 1.

23. $2\tan x + e^{\sec x} \cdot \sec x \tan x$.

24. $\dfrac{x^2 - ay}{ax - y^2}$.

25. $-\dfrac{2xy^2 + 3x^2}{2x^2y + 3y^2}$.

26. $\dfrac{3x^2+1}{3y^2+1}$.

27. $\dfrac{y^2 + 2xy - 1}{1 - 2xy - x^2}$.

28. $\dfrac{4\cos(2\log x^2 - 7)}{x}$.

29. For all values.

32. x, y are determined from $a^2y = \pm b^2x$ and equation of curve.

33. $x = \kappa\pi \pm \dfrac{\pi}{4}$.

34. $\tan^{-1} 2\sqrt{2}$.

Pages 54–55. Exercises on Chapter III

1. $72x$.

2. 0.

3. $-\dfrac{3!}{x^4}$.

4. $-\dfrac{5!}{x^6}$.

5. e^x.

6. $e^x \log x + \dfrac{2e^x}{x} - \dfrac{e^x}{x^2}$.

7. $2\log x + 3$.

8. $8\tan x \sec^2 x(3\sec^2 x - 1)$.

9. $2\cot x \csc^2 x$.

10. $16 \sin x \cos x$.

11. $\dfrac{24}{(1-x)^5}$.

12. $\dfrac{48}{x}$

13. $\sin x$.

14. $-\dfrac{8(e^x - e^{-x})}{(e^x + e^{-x})^3}$.

15. $8x^2e^{2x}$.

16. $-\dfrac{4!}{x^2}$.

17. a^ne^{ax}.

18. $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$.

19. $m^n \cdot \cos\left(mx + n\dfrac{\pi}{2}\right)$.

20. $\dfrac{(-1)^n \cdot (m+n-1)!}{(m-1)!(a+x)^{m+n}}$.

21. $\dfrac{m(-1)^{n-1} \cdot (n-1)!}{(a+x)^n}$.

22. $\dfrac{3p^3}{y^5}$.

23. $\dfrac{-b^4}{a^2y^3}$.

24. $\dfrac{-2a^3xy}{(y^2 - ax)^3}$.

25. $\dfrac{-y[(x-1)^2 + (y-1)^2]}{x^2(y-1)^3}$.

26. $\dfrac{3-y}{(2-y)^3} \cdot e^{2y}$.

32. $(-1)^n \cdot 2^{n-1} \cdot n!$
$$\left\{\frac{1}{(2x-1)^{n+1}} - \frac{1}{(2x+1)^{n+1}}\right\}.$$

33. $\dfrac{(n-1)!}{x}$.

34. $\dfrac{2(-1)^n \cdot n!}{(1+x)^{n+1}}$.

36. $2^{n-1}\cos\left(2x + \dfrac{n\pi}{2}\right)$.

Page 64. Art. 40

3. $e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \cdots$.

4. $-3 + 4(x-1) + (x-1)^2 + (x-1)^3$.

5. $-8 + 4(y-3) + 3(y-3)^2$.

Page 67. Art. 41

1. $x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + R$.

6. $1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + R$.

7. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + R$.

8. $1 - \frac{x^2}{2} - \frac{x^4}{8} + R$.

9. $1 + 2x + 2x^2 + 2x^3 + R$.

10. $x^4 + 7x^3 + 11x^2 - 16x - 41$.

Pages 75–76. Exercises on Chapter IV

1. $\cos x - h\sin x - \frac{h^2}{2}\cos x + \frac{h^3}{3!}\cdot\sin x + R$.

2. $\tan h + x\sec^2 h + x^2\sec^2 h\tan h + \frac{x^3}{3}\sec^2 h\,(1 + 3\tan^2 h) + R$.

4. $\log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + R$.

5. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

Pages 79–80. Art. 47

2. $\frac{2a^2}{a^2+b^2}$.

3. $-\frac{13}{7}$.

4. $\frac{1}{2a}$.

Pages 83–84. Art. 49

3. $\frac{2}{3}$.

4. 4.

5. $\frac{2}{5}$.

7. $\frac{1}{4}$.

8. -4.

9. 0.

10. 2.

11. 3.

12. $\frac{\log a}{\log b}$.

13. $\frac{a}{b}$.

14. $\frac{1}{n}$.

15. 1.

16. 1.

17. 1

18. $-\frac{2}{3}$.

19. $\frac{1}{2}$.

Pages 87–88. Art. 52

1. 1.

2. 0.

3. 0.

4. 5.

5. $\log a$.

6. $-\frac{5}{3}$.

7. $\frac{4a^2}{\pi}$.

8. 1.

9. 0.

10. 0 or ∞ according as $n > 0$, or < 0.

11. $\frac{1}{2}$.

12. $-\frac{1}{2}$.

13. $\frac{1}{3}$.

14. -1.

Page 89. Art. 54

1. 1.
2. $e^{-\frac{a^2}{2c^2}}$.
3. e^{ab}.
4. 1.
5. 1.
6. $e^{\frac{2}{\pi}}$.
7. 1.
8. 1.
9. e^a.

Pages 89-90. Exercises on Chapter V

1. 1.
2. 0.
3. ∞ if $r > 1$,
   0 if $r < 1$,
   me^{ma} if $r = 1$.
4. 1.
5. 0.
6. 0.
7. $\frac{a}{2}$.
8. 1.
9. $a_1 a_2 \cdots a_n$.
10. 0.
11. $-\frac{1}{2}$.
12. a.
13. $\frac{1}{\sqrt{2}\,b}$.
14. $e^{\frac{1}{3}}$.
15. $-\frac{1}{2\sqrt{2}}$.
16. -1.
17. 1.
18. $\frac{1}{2}$.

Page 100. Art. 62

1. $-\frac{1}{\sqrt{3}}$, max.; $\frac{1}{\sqrt{3}}$, min.
2. 2, max.; 3, min.
3. 2, min.; $\frac{8}{5}$, max.
4. $(2n + \frac{1}{4})\pi$, max.; $(2n + \frac{5}{4})\pi$, min. for all integral values of n.
5. $\frac{a}{4}$, min.
6. -1, max.; $-\frac{1}{3}$, min.
7. -2, min.; 1, max.
8. e, max.
9. $2n\pi$, min.; also $\tan^{-1} \pm \sqrt{\frac{2}{3}}$ for angles in 2d and 3d quarter. $(2n + 1)\pi$, $\tan^{-1} \pm \sqrt{\frac{2}{3}}$, 1st and 4th quarter, max.

Pages 103-104. Exercises on Chapter VI

1. The line should be bisected at the given point.
2. The altitude is equal to the diameter of the base.
3. The side parallel to the wall is double each of the others.
4. The diameter of sphere = edge of cube.
5. Three-fourths slant height of cone.
6. Area is $\frac{ab}{2}$.
7. $5\sqrt{\frac{1}{2}}$ inches.
8. 3 inches.
9. $\frac{2r}{\sqrt{3}}$.
10. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.
11. $\frac{a}{\sqrt{3}}$.
12. Arc $= 2\pi r(1 - \sqrt{\frac{2}{3}})$.
13. Circular arc is double the radius.
14. $\frac{Dr^{\frac{3}{2}}}{r^{\frac{3}{2}} + R^{\frac{3}{2}}}$, D being the distance between the centers of the spheres.
15. Angle at center of variable circle defined by $\theta = \cot\theta$.

Page 111. Exercises on Chapter VII

3. .00145.

6. $\frac{10}{\sqrt{2}}$ miles per hour.

7. The point (3, 6).

8. At 60°.

9. $2\,ab$.

10. ± 2.

11. $5\,\pi$.

12. 2.

13. 1 and 5.

16. $s = \frac{v_0^2}{64}$, $t = \frac{v_0}{32}$.

17. $\frac{a}{16}$.

19. ± 16, ∓ 12 feet per second.

Page 116. Art. 68

3. 1.

4. $(x + y)\cos xy$.

5. 1.

Page 120. Art. 71

3. $-\frac{ax + hy + g}{hx + by + f}$.

4. $\frac{x^3}{y^3}$.

Pages 127–128. Art. 76

2. $\frac{d^2x}{dy^2} - 2\,y\frac{dx}{dy} = 0$.

3. $R = -\frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$.

4. $\left(\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2}\right) = 0$.

5. $\frac{d^2z}{dx^2} = \cos^2 z + 2\left(\frac{dz}{dx}\right)^2$.

6. $\frac{d^2u}{dy^2} + u = 0$.

8. $\frac{d^2y}{du^2} + y = 0$.

9. $\frac{d^2y}{dt^2} = 0$.

11. $\frac{d^2x}{dy^2} + \frac{dx}{dy} + y = 0$.

12. $2\,z\frac{d^2z}{dx^2} + 2\left(\frac{dz}{dx}\right)^2$

$+ (1 - z^2)\,2\,z\frac{dz}{dx} + z^4 = 0$.

Pages 131–133. Art. 79

1. $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$,

$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$.

2. $y = x$.

3. $2\,y = 9\,x - 3$, $9\,y + 2\,x = 29$.

4. (α) $x + 2\,y = 4\,a$,

$y - 2\,x + 3\,a = 0$.

(β) $2\,y = \pm(x + 1)$,

$y = \mp 2\,x \pm 3$.

(γ) $y = x + p$, $x + y - 3\,p = 0$.

5 3.

6. $4\sqrt{17}$.

7. (α) Parallel at points of intersection with $ax + hy = 0$.
Perpendicular at points of intersection with $hx + by = 0$.

(β) Parallel at $\left(\frac{a}{-\sqrt[3]{2}}, \frac{3\,a\sqrt[3]{2}}{2}\right)$; perpendicular at $x = 0$.

(γ) Parallel at $\left(\frac{4\,a}{3}, \frac{2\sqrt[3]{4}\,a}{3}\right)$; perpendicular at (0, 0); (2 a, 0).

8. $\frac{1}{a}-\frac{1}{b}=\frac{1}{a'}-\frac{1}{b'}$, *i.e.* they must be confocal.

9. $\frac{\pi}{2}$. 11. $\frac{\pi}{2}$. 12. $\frac{y^2}{nx}$.

13. $\frac{2c+a}{3}$. 19. $(2p, \pm 2p\sqrt{2})$.

Pages 136–137. Art. 82

1. $\psi = \theta$.

2. Polar subtangent $= \frac{\rho^2}{a}$, Polar normal $= \sqrt{a^2+\rho^2}$, Polar subnormal $= a$

3. $\psi = \frac{\pi}{2} + 2\theta$, Subtangent $= -\rho \cot 2\theta$, Tangent $= \frac{a^2\rho}{\sqrt{a^4-\rho^4}}$,
Subnormal $= -\frac{a^2 \sin 2\theta}{\rho}$, Normal $= \frac{a^2}{\rho}$.

5. $\frac{\theta}{2}$, $2a\sin^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}$.

7. They have a common tangent at the pole ; elsewhere, $\frac{\pi}{2}$.

Page 142. Exercises on Chapter XI

1. $\sqrt{\frac{a+x}{x}}$, $2\sqrt{ax}$, $4\pi\sqrt{a^2+ax}$, $4\pi ax$. 2. $\frac{a}{y}$, $\frac{a}{x}$.

3. $\sec x$. 5. $\pi\frac{b^2}{a^2}(a^2-x^2)$. 7. $\sqrt{2a\rho}$.

4. πa^2x^2. 6. $\rho\sqrt{1+(\log a)^2}$. 8. $\frac{a^2}{\rho}$.

Page 151. Exercises on Chapter XII

1. $y=0$, $x=a$, $x=-a$.
2. $x=0$, $x=2a$, $y=a$, $y=-a$.
3. $y=a$, $y=-a$; two imaginary.
4. $y=a$; $x=c$ twice.
5. $y=-x+\frac{a}{3}$; two imaginary.
6. $x=1$; one parabolic branch.
7. $x=-a$, $y=-b$, $y=x+b-a$.
8. $x=0$ twice ; one parabolic branch.
9. $x=0$, $y=0$, $x+y=0$.
10. $y=x$; two imaginary.
11. $x+y+a=0$; two imaginary.
12. $y+x=0$; two imaginary.
13. $x=0$ twice ; $x=y$, $x=-y$.
14. $y=x$, $y=-x$; two imaginary.
15. $x+2y=0$, $x+y=1$, $x-y=-1$.

Page 158. Exercises on Chapter XIII

1. An inflexion at $x=y=2$.

3. $\left(\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$; $\left(\frac{-2a}{\sqrt{3}}, \frac{3a}{2}\right)$. 6. $x-4a^2y=0$.

8. Point of inflexion at $(a, \frac{4}{3}a)$, tangent is $x+y=\frac{7a}{3}$. Bending changes from negative to positive.

Pages 162–163. Art. 102

1. First.
2. They do not touch.
3. Third.
4. $3x(x-a)=a(y-a)$.
5. $y+12x=10$.
6. Second.
7. $a=-1$.

Page 168. Art. 108

1. $2a$.
2. $\frac{a}{2}$.
3. ∞.
4. $\frac{(x^2+y^2)^{\frac{3}{2}}}{2m^2}$.
5. $\frac{(e^2x^2-a^2)^{\frac{3}{2}}}{ab}$.
6. $\frac{(x^2+n^2y^2)^{\frac{3}{2}}}{n(n-1)xy}$.
7. $\frac{2(x+y)^{\frac{3}{2}}}{\sqrt{a}}$.
8. $3(axy)^{\frac{1}{3}}$.
9. $\frac{a\sqrt{x(8a-3x)^3}}{3(2a-x)^2}$.
10. $\frac{y^2}{a}$.

Page 170. Art. 109

1. $\rho\sqrt{1+(\log a)^2}$.
2. $\frac{a^2}{3\rho}$.
3. $\frac{a(5-4\cos\theta)^{\frac{3}{2}}}{9-6\cos\theta}$.
4. $\frac{2\rho^{\frac{3}{2}}}{\sqrt{a}}$.
5. $-\frac{\rho^3}{a^2}$.
6. $\frac{4\sqrt{\rho a}}{3}$.
7. $\frac{a(1+\theta^2)^{\frac{3}{2}}}{\theta^4}$.

Page 176. Art. III

1. $\alpha=0,\ \beta=0$.
2. $\alpha=a\log\frac{a+\sqrt{a^2-y^2}}{y},\ \beta=\frac{a^2}{y}$.
3. $\alpha=\frac{a^4+15y^4}{6a^2y},\ \beta=\frac{a^4y-9y^5}{2a^4}$.
4. $\alpha=x-\frac{y}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right),\ \beta=2y$.
5. $(\alpha+\beta)^{\frac{2}{3}}-(\alpha-\beta)^{\frac{2}{3}}=(4a)^{\frac{2}{3}}$.
6. $(a\alpha)^{\frac{2}{3}}-(b\beta)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$.
7. $(\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}}=2a^{\frac{2}{3}}$.

Page 186. Exercises on Chapter XV

1. (0, 0); $ax\pm by=0$.

2. (0, 0); cusp of first kind, $y=0$.

3. Four cusps of first kind; $(0,\pm a), (\pm a, 0)$; $y=0, x=0$.

4. (0, 0); conjugate point with real coincident tangents, $y=0$.

5. $(0, a)$; $y=a+x$; cusp of second kind.

6. Two nodes at infinity; the asymptotes are $x=y\pm 1, x+y=\pm 1$.

7. $(0,-a)$; $(+a, 0)$; $(-a, 0)$: the tangents are, respectively,

$$\sqrt{3}(y+a)=\pm\sqrt{2}x;$$
$$2(x+a)=\pm\sqrt{3}y;$$
$$2(x-a)=\pm\sqrt{3}y.$$

8. $(-a, 0)$; conjugate point.

9. (0, 0); $x=0$, $y=0$.

10. (0, 0) is a tacnode; $y=0$.

Pages 193–194. Exercises on Chapter XVI

1. $x^2 + y^2 = p^2$.
2. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.
4. $4\,xy = k^2$.
5. $(x - \alpha)^2 + (y - \beta)^2 = r^2$.
6. $y^2(x + 2\,a) + x^3 = 0$.
7. $y^2 = 4\,a(2\,a - x)$.
8. (c, c), $(0, 0)$, and one at infinity on each axis.

Page 201. Art. 125

1. $\frac{2}{3}x^{\frac{3}{2}}$.
2. $\dfrac{x^{a+1}}{a+1}$.
3. $\frac{3}{2}x^{\frac{2}{3}}$.
4. $\dfrac{2\,m}{2\,m - 1}x^{m-\frac{1}{2}}$.
5. $ax - \frac{9}{4}a^{\frac{2}{3}}x^{\frac{4}{3}} + \frac{9}{5}a^{\frac{1}{3}}x^{\frac{5}{3}} - \frac{1}{2}x^2$.
6. $5\log x + \dfrac{3}{2\,x^2} - \dfrac{1}{3\,x^3}$.
7. $\frac{1}{6}(x^2 + a^2)^3$.
8. $\dfrac{(ax + b)^{n+1}}{a(n + 1)}$.
9. $\log(x + a)$.
10. $\log\sqrt{2\,ax - x^2}$.
11. $\log\tan x$.
12. $-\log(1 + \cos x)$.
13. $\log(\log x)$.
14. $\frac{5}{3}\log(x^3 + 1)$.
15. $\log\sec x$.
16. $\log\sin x$.
17. $\dfrac{1}{a}e^{ax}$.
18. $\frac{1}{2}e^{x^2}$.
19. $\dfrac{(a + b)^{m+nx}}{n\log(a + b)}$.
20. $\frac{1}{2}(x + \sin x)$.
21. $-\dfrac{\cos(m + n)x}{m + n}$.
22. $-\frac{1}{2}\cos x^2$.
23. $\sin x - \frac{1}{3}\sin^3 x$.
24. $-\cos x + \frac{1}{3}\cos^3 x$.
25. $\frac{1}{2}x - \frac{1}{4}\sin 2\,x$.
26. $\tan x - x$.
27. $\frac{1}{3}\tan^3 x$.
28. $-\dfrac{1}{a}\cot(ax + b)$.
29. $-\frac{2}{3}(\cot x)^{\frac{3}{2}}$.
30. $\log\tan x$.
31. $\frac{1}{3}\sec^3 u$.
32. $-\log\operatorname{cosec} u$.
33. $-\cos u$.
34. $\sin^{-1}\dfrac{x}{a}$.
35. $\frac{1}{2}\sin^{-1}2\,x$.
36. $\dfrac{1}{a}\tan^{-1}\dfrac{u}{a}$.
37. $\dfrac{1}{ab}\tan^{-1}\dfrac{bu}{a}$.
38. $\frac{1}{2}\tan^{-1}\dfrac{x - 1}{2}$.
39. $\sec^{-1}(x - 1)$.
40. $\dfrac{1}{a}\sec^{-1}\dfrac{x}{a}$.

Page 205. Art. 126

1. $x\sin^{-1}x + \sqrt{1 - x^2}$.
2. $e^x\tan^{-1}e^x - \frac{1}{2}\log(1 + e^{2x})$.
3. $x^2\sin x + 2\,x\cos x - 2\sin x$.
4. $\dfrac{x^{n+1}}{n+1}\left(\log x - \dfrac{1}{n+1}\right)$.
5. $\frac{1}{6}[2\,x^3\tan^{-1}x - x^2 + \log(1 + x^2)]$.
6. $\sec x\,[\log\cos x + 1]$.
7. $\frac{1}{2}[(x^2 + 1)\cot^{-1}x + x]$.
8. $\frac{1}{9}[\sin 3\,x - 3\,x\cos 3\,x]$.
9. $\frac{1}{2}e^x(\sin x + \cos x)$.
10. $\frac{1}{2}e^x(\sin x - \cos x)$.
11. $\frac{2}{3}\cos x\sin 2\,x - \frac{1}{3}\cos 2\,x\sin x$.

Pages 206–209. Art. 127

4. $\frac{1}{2}(\sin^{-1}x)^2$.

6. $\frac{1}{2}\cos(x^2+1)[1-\log\cos(x^2+1)]$.

8. $\frac{1}{a}\tan^{-1}\frac{x}{a}$.

9. $\frac{1}{a}\sec^{-1}\frac{x}{a}$, or $\frac{1}{a}\cos^{-1}\frac{a}{x}$.

10. $\operatorname{vers}^{-1}\frac{x}{a}$, or $\sin^{-1}\frac{x-a}{a}$.

11. $\log(x+\sqrt{x^2\pm a^2})$.

12. $\frac{1}{2a}\log\frac{x-a}{x+a}$.

13. $\log\tan\frac{x}{2}$.

14. $\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$.

16. $\frac{1}{\sqrt{5}}\tan^{-1}\frac{2x+1}{\sqrt{5}}$.

17. $\frac{1}{\sqrt{14}}\tan^{-1}\frac{3x-1}{\sqrt{14}}$.

18. $-\frac{1}{3}\log\frac{x-2}{x+1}$.

19. $\frac{1}{3}\sin^{-1}(3x-5)$.

20. $\frac{1}{3}\cos(3x-2)$
$+(x-\frac{2}{3})\sin(3x-2)$.

21. $\log\frac{\sqrt{a^2+bx}-a}{\sqrt{a^2+bx}+a}$.

Pages 213–214. Exercises on Chapter I

1. $\sqrt{x^2+2x+2}-\log(x+1+\sqrt{x^2+2x+2})$.

2. $-\log\frac{1-2x+\sqrt{5x^2-4x+1}}{x}$.

3. $\frac{2}{3}\sqrt{3x^2+x-2}-\frac{10}{3\sqrt{3}}\log(x+\frac{1}{6}+\sqrt{x^2+\frac{1}{3}x-\frac{2}{3}})$.

4. $-\sqrt{8+4x-4x^2}+\frac{7}{2}\sin^{-1}\frac{2x-1}{3}$.

5. $-\sqrt{-x^2+2x+1}+\sin^{-1}\frac{x-1}{\sqrt{2}}$.

6. $\sqrt{1-x^2}+\sin^{-1}x$.

7. $\sqrt{ax-x^2}+\frac{a}{2}\sin^{-1}\frac{2x-a}{a}$.

8. $\sqrt{x^2+2x+3}-\log(x+1+\sqrt{x^2+2x+3})-\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}+\sqrt{x^2+2x+3}}{x+1}\right]$.

9. $\sqrt{x^2+x+1}-\frac{1}{2}\log[x+\frac{1}{2}+\sqrt{x^2+x+1}]-\log\left[\frac{1-x+2\sqrt{x^2+x+1}}{2(x+1)}\right]$.

10. $\log\frac{1}{x}(x^2+1+\sqrt{x^4+x^2+1})$.

11. e^{e^x}.

12. $\frac{5}{8}\tan^{-1}\frac{x^4}{2}$.

13. $\frac{1}{6}\log(6x^3+12x+5)$.

14. $-\log(e^{-x}+\sqrt{e^{-2x}-1})$.

15. $-\frac{1}{2\sqrt{2}}\log\left[\frac{\sqrt{2}+\sqrt{x^2+2}}{x}\right]$.

16. $\frac{1}{2}\sin^{-1}x^2$.

17. $\frac{1}{2}\log\frac{e^x-1}{e^x+1}$.

18. $\frac{1}{5}x^5\tan^{-1}x-\frac{1}{20}x^4+\frac{1}{10}x^2-\frac{1}{10}\log(x^2+1)$.

19. $x-\log(x^2+1)+2\tan^{-1}x$.

20. $-\frac{1}{a^x(\log a)^3}[2 + 2\,x \log a + (x \log a)^2]$.

21. $\tan\theta - \sec\theta$.

22. $-\cot\frac{\theta}{2}$.

23. $\frac{1}{2(b-a)}\log(a\cos^2 x + b\sin^2 x)$.

24. $\frac{1}{2}[x - \log(\sin x + \cos x)]$.

25. $\frac{1}{3}(x^2 + 2)\sqrt{x^2 - 1}$.

26. $6[\frac{1}{7}x^{\frac{7}{6}} + \frac{1}{6}x - \frac{1}{5}x^{\frac{5}{6}} + \frac{1}{4}x^{\frac{2}{3}} - \frac{1}{3}x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{3}} - x^{\frac{1}{6}} + \log(x^{\frac{1}{6}} + 1)]$.

27. $-\frac{1}{3}\sqrt{1 - \log x}$.

28. $\log\sqrt{e^{2x} + 1}$.

29. $\sin^{-1}\left(\frac{2\sin x + 1}{3}\right)$.

30. $\tan^{-1}(\log x)$.

31. $\frac{1}{b(a - b\tan x)}$.

32. $\frac{1}{2}\sin^{-1}\left[\frac{2(x-a)^2 + a^2}{\sqrt{5}\,a^2}\right]$.

33. $2\sqrt{x}\,\text{vers}^{-1}\frac{x}{a} + 4\sqrt{2\,a - x}$.

34. $-\frac{1}{x}\sqrt{3\,x^2 + 2\,x + 1} + \log\left[\frac{x + 1 + \sqrt{3\,x^2 + 2\,x + 1}}{x}\right]$.

35. $-\frac{1}{2\,x^2}\log(x + \sqrt{x^2 - a^2}) + \frac{1}{2}\frac{\sqrt{x^2 - a^2}}{a^2x}$.

Pages 219–222. Exercises on Chapter II

2. $\frac{x-1}{2}\sqrt{x^2 - 2\,x - 3} - 2\log(x - 1 + \sqrt{x^2 - 2\,x - 3})$.

3. $\frac{x-a}{2}\sqrt{2\,ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}$.

5. The arrangements which can be used are [*B*], [*C*], [*B*], [*C*], and [*B*], [*B*], [*C*], [*C*].

6. $\frac{1}{8}\left[\frac{x}{x^2 + 4} + \frac{1}{2}\tan^{-1}\frac{x}{2}\right]$.

7. $\frac{2\,x - 1}{3(x^2 - x + 1)} + \frac{4}{3\sqrt{3}}\tan^{-1}\frac{2\,x - 1}{\sqrt{3}}$.

8. $-\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$.

9. $\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$.

10. $-\frac{\sqrt{a^2 - x^2}}{a^2x}$.

11. $\frac{x}{3\,a(x^2 + a)^{\frac{3}{2}}} + \frac{2\,x}{3\,a^2\sqrt{x^2 + a}}$.

12. $\frac{x}{8}(2\,x^2 + 5\,a^2)\sqrt{x^2 + a^2} + \frac{3\,a^4}{8}\log(x + \sqrt{x^2 + a^2})$.

13. $\frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2}\log(x + \sqrt{x^2 + a})$.

14. $\frac{1}{6}(2\,x^2 - ax - 3\,a^2)\sqrt{2\,ax - x^2} + \frac{a^3}{2}\sin^{-1}\frac{x - a}{a}$.

15. $-\frac{(2\,ax - x^2)^{\frac{3}{2}}}{3\,ax^3}$.

16. $-\frac{\sqrt{1 + x^4}}{2\,x^2}$.

17. $\dfrac{3(x+2)^3-5(x+2)}{8(x^2+4x+3)^2}+\dfrac{3}{16}\log\dfrac{x+1}{x+3}.$

18. $\frac{1}{2}(x+1)\sqrt{1-2x-x^2}+\sin^{-1}\dfrac{x+1}{\sqrt{2}}.$

Page 226. Art. 133

2. $\dfrac{1}{2a}\log\dfrac{x-a}{x+a}.$

3. $\log\dfrac{(x+1)^2}{x(x-1)}.$

4. $\dfrac{x^2}{2}-4x+\dfrac{13}{2}\log\dfrac{(x+3)^3}{x+1}.$

5. $x+\log(x-a)^a(x-b)^b.$

6. $\dfrac{2+\sqrt{3}}{2\sqrt{3}}\log(x-2-\sqrt{3})-\dfrac{2-\sqrt{3}}{2\sqrt{3}}\log(x-2+\sqrt{3}).$

7. $\frac{1}{20}\log\dfrac{(2x-1)(x-2)}{(2x+1)(x+2)}.$

8. $\log\dfrac{(x-a)(x-b)}{x-c}.$

9. $x+\dfrac{1}{b-a}[a^2\log(x+a)-b^2\log(x+b)].$

10. $\log[(x+2)\sqrt{2x-1}].$

11. $\log\dfrac{(x-a)(x+b)}{x}.$

12. $\frac{1}{3}\log\dfrac{x^6}{(2+x)(1-x)^5}.$

13. $\dfrac{x^2}{2}-7x+64\log(x+4)$

$-27\log(x+3).$

14. $\dfrac{1}{2ab}\log\dfrac{ax-b}{ax+b}.$

15. $\sin^{-1}\dfrac{x}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{(x-1)\sqrt{2}}{x-2}-\log(x-1)^{-1}(2-x+\sqrt{2-x^2}).$

16. $\log(x+2+\sqrt{x^2+4x+7})-\dfrac{1}{\sqrt{3}}\log(x+2)^{-1}(\sqrt{3}+\sqrt{x^2+4x+7})$

$+\log(x+3)^{-1}(1-x+2\sqrt{x^2+4x+7}).$

Page 227. Art. 134

2. $-\dfrac{1}{2(x-1)}+\frac{1}{4}\log\dfrac{x+1}{x-1}.$

3. $x-5\log(x-3)\dfrac{-2}{x-3}.$

4. $\dfrac{1}{2(a^2-x^2)}.$

5. $\dfrac{-1}{\sqrt{2}x(x+\sqrt{2})}.$

6. $\dfrac{x^2}{2}-2x+\dfrac{2x+3}{x^2+x}+\log x(x+1)^2.$

7. $\log(x^2-a^2)-\dfrac{2a^2}{x^2-a^2}.$

8. $\dfrac{1}{4(\sqrt{2}+1-x)^2}.$

9. $ax-\dfrac{1}{x}+\log\dfrac{x}{x+a}.$

10. $x+\dfrac{1}{3x}-\frac{1}{9}[28\log(x+3)-\log x].$

11. $\dfrac{b}{a^2}\log\dfrac{a+bx}{x}-\dfrac{1}{ax}.$

12. $2\log[x-1+\sqrt{x^2-2x+2}]$

$-\log\left[\dfrac{1+\sqrt{x^2-2x+2}}{x-1}\right]$

$+\dfrac{x}{x-1}\sqrt{x^2-2x+2}.$

13. $\log(x-a)+\dfrac{3a^2-4ax}{2(x-a)^2}.$

Page 228. Art. 135

2. $\log\dfrac{x}{\sqrt{x^2+4}}$.

3. $\frac{1}{4}\log\dfrac{x^2+1}{(x+1)^2}+\frac{1}{2}\tan^{-1}x$.

4. $\dfrac{1}{3\,a^2}\Big[\log(x+a)$
$-\frac{1}{2}\log(x^2-ax+a^2)$
$+\sqrt{3}\tan^{-1}\dfrac{2x-a}{a\sqrt{3}}\Big]$.

5. $-\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+\dfrac{1}{b}\tan^{-1}\dfrac{x}{b}$.

6. $\dfrac{3}{x}+\log\dfrac{x^2}{x^2+2}$
$+\dfrac{3}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}$.

7. $-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{\sqrt{3}}{2\,x^2+1}$.

8. $-\dfrac{1}{2\,a(x-a)}-\dfrac{1}{2\,a^2}\tan^{-1}\dfrac{x}{a}$.

9. $x-\log\dfrac{x^2+2\,x+2}{x-1}$.

Page 230. Art. 136

2. $\tan^{-1}x+\dfrac{1}{x^2+1}$.

3. $\dfrac{3}{2\,a}\tan^{-1}\dfrac{x}{a}+\dfrac{x-2\,a}{2\,(x^2+a^2)}$.

4. $\frac{1}{4}\log\dfrac{x^2+1}{(x+1)^2}+\dfrac{x-1}{2(x^2+1)}$.

5. $-\dfrac{2}{x}+3\log\dfrac{\sqrt{x^2+1}}{x}$
$-\dfrac{3+2\,x}{2\,(x^2+1)}-3\tan^{-1}x$.

6. $\dfrac{x}{2\,(x^2+a^2)}+\log(x^2+a^2)$
$-\dfrac{1}{2\,a}\tan^{-1}\dfrac{x}{a}$.

7. $\dfrac{1}{x^2+1}-\dfrac{1}{4(x^2+1)^2}+\frac{1}{2}\log(x^2+1)$.

Page 232. Art. 138

2. $-2\sqrt{x}-3\,x^{\frac{1}{3}}+12\,x^{\frac{1}{6}}+6\log(x^{\frac{1}{3}}+1)-12\tan^{-1}x^{\frac{1}{6}}$.

3. $\log\dfrac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$.

4. $2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log(\sqrt[6]{x}+1)$.

5. $2\log(\sqrt{x-1}+1)-\dfrac{2}{\sqrt{x-1}+1}$.

6. $2\tan^{-1}\sqrt{x-2}$.

7. $\dfrac{1}{b}\log\dfrac{\sqrt{x-a}-b}{\sqrt{x-a}+b}$.

8. $14(x^{\frac{1}{14}}-\frac{1}{2}\,x^{\frac{1}{7}}+\frac{1}{3}\,x^{\frac{3}{14}}-\frac{1}{4}\,x^{\frac{2}{7}}+\frac{1}{5}\,x^{\frac{5}{14}})$.

Page 236. Art. 139

3. $2\log\left[1-\sqrt{\dfrac{1+x}{1-x}}\right]+\dfrac{2(x-1)}{x-1+\sqrt{1-x^2}}$.

4. $-2\log\left[\sqrt{2}+\sqrt{\dfrac{2\,x-1}{x-1}}\right]$.

Pages 236–237. Exercises on Chapter IV

1. $\dfrac{\sqrt{x^2-1}}{x^2+2}$.

2. $\frac{3}{2}(x-a)^{\frac{1}{3}}-\dfrac{3}{4\sqrt{2}}\log\dfrac{\sqrt{2}(x-a)^{\frac{1}{3}}-1}{\sqrt{2}(x-a)^{\frac{1}{3}}+1}$.

3. $\dfrac{-x}{2(x^2+1)\sqrt{x^2+2}}$.

4. $\log(x+\sqrt{x-1})-\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\sqrt{x-1}+1}{\sqrt{3}}$.

5. $\frac{1}{2}[x^2-x\sqrt{x^2-1}+\log(x+\sqrt{x^2-1})]$.

6. $\frac{1}{2}x(\sqrt{x^2+2}-\sqrt{x^2+1})+\frac{1}{2}\log\dfrac{x+\sqrt{x^2+2}}{x+\sqrt{x^2+1}}$. $\to \frac{1}{2}\ln\left(x+\sqrt{x^2+2}-\frac{1}{2}\ln\right.$

7. $\frac{3}{10}(2x-3a)(a+x)^{\frac{2}{3}}$.

8. $6\log(x^{\frac{1}{3}}-3x^{\frac{1}{6}}+5)$.

9. $\dfrac{-4}{\sqrt{\sqrt{x+1}+\sqrt{x-1}}}$.

10. $\dfrac{1}{2\sqrt{2}\,a^2}\log\dfrac{\sqrt{x^2-a^2}+x\sqrt{2}}{\sqrt{x^2-a^2}-x\sqrt{2}}$.

11. $\frac{6}{7}x^{\frac{7}{6}}-\frac{6}{5}x^{\frac{5}{6}}+\frac{3}{2}x^{\frac{2}{3}}+2x^{\frac{1}{2}}-3x^{\frac{1}{3}}-6x^{\frac{1}{6}}+3\log(x^{\frac{1}{3}}+1)+6\tan^{-1}x^{\frac{1}{6}}$.

12. $-\frac{3}{8}\left(\dfrac{1-x}{1+x}\right)^{\frac{4}{3}}$.

Page 239. Art. 142

1. $\frac{1}{3}\tan^3 x+\tan x$.
2. $-\frac{1}{3}\cot^3 x-\cot x$.
3. $\tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x$.
4. $-128[\cot 2x+\cot^3 2x$
   $+\frac{3}{5}\cot^5 2x+\frac{1}{7}\cot^7 2x]$.
5. $\frac{2}{3}\operatorname{cosec}^3 x-\cot x-\frac{2}{3}\cot^3 x$.
6. $-64[\cot 4x+\frac{1}{3}\cot^3 4x]$.
7. $-\dfrac{1}{2\tan^2 x}+\log\tan x$.
8. $-\frac{1}{3}\cot^3 x-\frac{1}{5}\cot^5 x$.

Page 240. Art. 143

1. $\frac{1}{4}\sec^4 x-\frac{1}{2}\sec^2 x$.
2. $-\frac{1}{7}\operatorname{cosec}^7 x+\frac{2}{5}\operatorname{cosec}^5 x$
   $-\frac{1}{3}\operatorname{cosec}^3 x$.
3. $\dfrac{1}{a}(\frac{1}{5}\sec^5 ax-\frac{2}{3}\sec^3 ax+\sec ax)$.
4. $-(\sin x+\operatorname{cosec} x)$.
5. $\frac{1}{4}\sec^4 x-\sec^2 x+\log\sec x$.
6. $\dfrac{\sec^{n-1}x}{n-1}-\dfrac{\sec^{n\;\;3}x}{n-3}$.
7. $\log\sec x$.
8. $-\log\operatorname{cosec} x$.

Page 241. Art. 144

1. $-\frac{1}{3}\cot^3 x+\cot x+x$.
2. $\dfrac{1}{2a}\tan^2 ax-\dfrac{1}{a}\log\sec ax$.
3. $\frac{1}{2}(\tan^2 x+\cot^2 x)$
   $+4\log(\sin x\cos x)$.
4. $\dfrac{\tan^{n-1}x}{n-1}$.
5. $\frac{1}{7}\tan^7 x-\frac{1}{5}\tan^5 x+\frac{1}{3}\tan^3 x$
   $-\tan x+x$.

Pages 242–244. Art. 145

2. $-\cos x + \frac{1}{3}\cos^3 x.$

3. $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x.$

4. $\log \sin x - \sin^2 x + \frac{1}{4}\sin^4 x.$

5. $\frac{3}{4}\cos^{-\frac{4}{3}} x + 3\cos^{\frac{2}{3}} x - \frac{3}{8}\cos^{\frac{8}{3}} x.$

6. $\frac{4}{3}(1-\cos x)^{\frac{3}{2}} - \frac{2}{5}(1-\cos x)^{\frac{5}{2}}.$

✓ 8. $-\frac{1}{3}\cot^3 x.$

9. $-\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x.$

10. $-\cot^5 x\left(\frac{1}{5} + \frac{1}{7}\cot^2 x\right).$

11. $-\frac{1}{3}\cot^3 x - 2\cot x + \tan x.$

12. $\frac{2}{3}\sqrt{\tan x}\,(\tan x - 3\cot x).$

13. $\frac{\tan^{n-1} x}{n-1} + \frac{\tan^{n+1} x}{n+1}.$

15. $\frac{1}{8}x - \frac{1}{32}\sin 4x.$

16. $\frac{1}{128}\left(5x + \frac{8}{3}\sin^3 2x - \sin 4x - \frac{1}{8}\sin 8x\right).$

17. $\frac{1}{128}\left(3x - \sin 4x + \frac{1}{8}\sin 8x\right).$

18. $\frac{1}{8}\left(3x + \sin 4x + \frac{1}{8}\sin 8x\right).$

19. $-\frac{1}{2}\cot x - \frac{1}{6}\cot^3 x.$

20. $\log \tan 2x.$

21. $\tan x + \frac{1}{4}\sin 2x - \frac{3}{2}x.$

22. $2\cot x - \frac{1}{3}\cot^3 x + \frac{5}{2}x + \frac{1}{4}\sin 2x.$

23. $-\frac{1}{3a^2}\left(\frac{\sqrt{a^2-x^2}}{x}\right)^3.$

24. $\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}.$

25. $\frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$

Pages 245–246. Art. 146

1. $\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right).$

2. $$-\frac{1}{\sqrt{a^2+b^2}} \times \log \frac{b\tan\frac{x}{2} - a - \sqrt{a^2+b^2}}{b\tan\frac{x}{2} - a + \sqrt{a^2+b^2}}.$$

3. $\frac{1}{4}\tan^{-1}\left(\frac{\tan x}{2}\right).$

4. $\frac{1}{2}\tan^{-1}\left[2\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right].$

5. $$\frac{1}{2\sqrt{3}}\log\frac{\sqrt{3}\tan\left(x-\frac{\pi}{4}\right)-1}{\sqrt{3}\tan\left(x-\frac{\pi}{4}\right)+1}$$
$$=\frac{1}{2\sqrt{3}}\log\frac{\tan x - 2 - \sqrt{3}}{\tan x - 2 + \sqrt{3}}.$$

6. $-\frac{1}{a(a\tan x + b)}.$

7. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right).$

Page 247. Exercises on Chapter V

3. $2\sqrt{\tan x}.$

4. $\sqrt{\frac{1+x}{1-x}}.$

5. $\frac{1}{4}\tan^2 x(2+\tan^2 x).$

6. $-\frac{1}{\sin x} + \log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right).$

7. $\frac{1}{2}\tan^2 x\sin x + \frac{3}{2}\left[\sin x - \log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right].$

8. $e^{\frac{x}{2}}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right).$

9. $-\frac{1}{2}e^{-x}(\sin x + \cos x).$

10. $\frac{1}{8}e^{2x}(2 - \sin 2x - \cos 2x).$

11. $\frac{1}{4}e^{x}\left(\sin x + \cos x - \frac{3}{5}\sin 3x - \frac{1}{5}\cos 3x\right).$

15. $\frac{\sin^{n+1} x}{n+1} - \frac{\sin^{n+3} x}{n+3}.$

16. $\frac{2}{3}\tan^{\frac{3}{2}} x - 2\sqrt{\cot x}.$

17. $-32\cot 2x(1+\frac{2}{3}\cot^2 2x+\frac{1}{5}\cot^4 2x)$.

18. $\frac{1}{4}\tan\frac{1}{2}x(1+\frac{2}{3}\tan^2\frac{1}{2}x+\frac{1}{5}\tan^4\frac{1}{2}x)$.

Page 259. Art. 152

5. $\frac{\pi}{2a}$. 6. 2. 7. ∞.

Pages 262–263. Art. 154

1. $\frac{4ab^3}{3}$. 2. $2ab$. 3. 2. 4. $20\sqrt{5a}$. 6. $2a$. 7. 12.

8. $\frac{1}{6}$. 9. $\frac{7}{6}$. 10. $\frac{8}{3}a\sqrt{ap}$; $\frac{8}{3}p^2$. 11. πa^2. 12. $4a^2\tan^{-1}\frac{x_1}{2a}$; $4\pi a^2$.

13. πab. 14. $\frac{3}{8}\pi a^2$. 15. $\frac{a^{x_2}-a^{x_1}}{\log a}$.

Page 266. Art. 155

4. π. 5. $3\pi a^2$. 6. $4a^2$. 7. ∞. 8. $\frac{24}{5}$. 9. $k<1$.

Pages 268–269. Art. 156

2. $\frac{3\pi a^2}{2}$.

3. πa^2.

4. $\frac{1}{2}a(\rho_1-\rho_2)$.

5. $\frac{4a^2}{3}$.

6. πa^2.

7. $\frac{\pi a^2}{2}$.

8. $\frac{\pi}{\sqrt{AB-H^2}}$.

Pages 270–271. Art. 157

1. $p[\sqrt{2}+\log(1+\sqrt{2})]$.

2. $\frac{61a}{216}$.

3. $6a$.

4. $2\pi r$.

5. $\frac{a}{2}(e^{\frac{x_1}{a}}-e^{-\frac{x_1}{a}})$.

6. $2-\sqrt{2}+\log\frac{1+\sqrt{2}}{\sqrt{3}}$.

7. $\frac{4(a^3-b^3)}{ab}$.

Page 272. Art. 158

2. $8a$.

3. $2a\left[\sqrt{5}-2-\sqrt{3}\log\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}(2+\sqrt{3})}\right]$.

4. $2\pi a$.

5. $a\left[\tan\frac{\theta}{2}\sec\frac{\theta}{2}+\log\left(\tan\frac{\theta}{2}+\sec\frac{\theta}{2}\right)\right]_{\theta_1}^{\theta_2}$.

6. $a\left[-\frac{1}{\theta}\sqrt{1+\theta^2}+\log(\theta+\sqrt{1+\theta^2})\right]_{\theta_1}^{\theta_2}$.

Pages 273–274. Art. 159

1. $8a$. 2. $\frac{8ma}{m-1}$ 3. $6a$. 4. $\frac{1}{2}a\theta_1^2$.

5. $\frac{1}{2}(x_1^{\frac{2}{3}}+y_1^{\frac{2}{3}})^{\frac{3}{2}}-\frac{a}{2}$. 7. $\sqrt{2}(e^{\theta_1}-1)$.

Pages 276-277. Art. 160

1. $4\pi a^2\left(1-\frac{1}{e}\right)$. **3.** $\frac{8}{3}\pi p^2(\sqrt{8}-1)$.

2. $\pi(\pi-2)$. **4.** $\frac{\pi p^2}{2}[3\sqrt{2}-\log(1+\sqrt{2})]$.

5. (*a*) $2\pi b\left(b+\frac{a^2}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b}{a}\right)$.

(*b*) $2\pi a^2+\frac{\pi a b^2}{\sqrt{a^2-b^2}}\log\left[\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right]$.

6. $\frac{32}{5}\pi a^2$. **7.** (α) $\pi b\sqrt{a^2+b^2}$; (β) $\pi a\sqrt{a^2+b^2}$. **9.** $\frac{64}{3}\pi a^2$.

8. $4\pi a^2$.

Pages 278-279. Art. 161

1. $\frac{4\pi a^2 b}{3}$. **3.** πk^3; ∞. **5.** $\frac{\pi a^3}{15}$.

2. $\frac{4\pi r^3}{3}$. **4.** $\frac{32\pi a^3}{105}$. **6.** $4\pi^2 a^3$.

7. $\pi\left[8a^3\log\frac{2a}{y_1}-4a^2(2a-y_1)\right]$; ∞.

8. $5\pi^2 a^3$. **9.** $\frac{8\pi a^3}{3}$. **10.** $\frac{\pi r^3}{6\sqrt{2}}(10-3\pi)$

Pages 280-283. Art. 162

1. $\frac{4}{3}\pi abc$. **2.** $\frac{1}{3}\pi ab$. **5.** $4\frac{1}{2}$ cu. ft. **7.** $\frac{4}{\sqrt{3}}$ cu. ft.

3. $\frac{1}{3}Ah$. **4.** $\frac{\pi abp}{2}$. **8.** $\frac{8}{3}a^3$. **9.** $\rho=e^{\frac{\theta}{a}+c}$.

11. $\frac{4}{3}\pi a^3\cos^4\theta$. **12.** $\frac{1}{2}\pi abc^2$.

Pages 284-285. Exercises on Chapter VII

1. $a^2\log n$. **2.** 1. **5.** $a^2\left(2+\frac{\pi}{2}\right)$. **6.** $\frac{8}{15}$.

3. $\frac{3\pi(a^2-b^2)^2}{8ab}$. **4.** $\frac{a^2}{6}$. **7.** $4ab\tan^{-1}\frac{b}{a}$. **9.** $\frac{\rho_2^2-\rho_1^2}{4a}$.

10. $\frac{a}{2}\left[\theta\sqrt{1+\theta^2}+\log(\theta+\sqrt{1+\theta^2})\right]_{\theta_1}^{\theta_2}$.

11. $4\pi^2 ak$. **12.** $2\pi^2a^2k$.

13. $\frac{\pi a^3}{4}[e^{\frac{2x_1}{a}}-e^{\frac{-2x_1}{a}}]+\pi a^2x_1$. **14.** $2\pi^2a^3$.

15. $2\pi a^2(3\sin t_1-3t_1\cos t_1-t_1^2\sin t_1)$.

16. $\frac{\pi a^3}{3}$. **17.** $2\pi a^2$. **18.** $a\log\frac{a}{y_1}$.

Pages 287–288. Art. 163

3. $xy = Cy^2 + C'y + \frac{1}{2}$. 4. $y = kx(\log x - 1) + C_1x + C_2$. 5. $\frac{4}{15}kt^{\frac{5}{2}}$.

Page 292. Art. 165

1. $xy + C$. 2. $-\cos x \cos y + C$. 5. $x^3 + y^3 - 3axy + C$.

3. Impossible. 4. $\log \frac{x}{y} + C$. 6. $\tan^{-1} \frac{x}{y}$.

7. $\frac{2}{3}x^3 + x^2y + 5x + \frac{1}{3}y^3 - \frac{1}{2}y^2 + C$.

8. $\frac{1}{5}x^5 + xy^4 + \frac{1}{3}x^3 - xy^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 + 2y + C$.

Page 294. Art. 167

1. 1. 2. $\frac{4}{3}a^3$. 3. $6b^3$. 4. $\frac{\pi}{2}$.

Page 295. Art. 168

2. 64. 3. $\frac{8\pi}{3} - 2\sqrt{3}$.

Page 298. Art. 169

2. $\frac{2a^3m}{3}$. 3. $\frac{16}{3}a^3$. 4. π. 5. $\frac{4}{3}\pi abc$. 6. $\frac{\pi ab}{2}$.

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